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Direct and adaptive approaches to multi-objective optimization

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Abstract: We formulate in a unified way the major theoretical results obtained by the authors in the domain of multi-objective differential optimization, discuss illustrative examples, and present a brief discussion of the related software developments made at Inria. The development is split in two connected parts. In Part A, the Multiple Gradient Descent Algorithm (MGDA), referred to as the direct approach, is a general construction of a descent method in the multi-objective optimization context. The algorithm provides a technique for determining Pareto optimal solutions in constrained problems as an extension of the classical steepest-descent method. In Part B, another problematics is posed, referred to as the adaptive approach. It is meant to be developed after a Pareto-optimal solution with respect to a set of primary cost functions subject to constraints has been elected in a first phase of optimization carried out by application of MGDA, or another effective multi-objective optimization technique, possibly an evolutionary strategy. This second phase of optimization permits to construct a continuum of neighboring solutions for which novel cost functions, designated as secondary cost functions, are reduced at the cost of a moderate degradation of the Pareto-stationarity condition of the primary cost functions. In this way, the entire optimization process demonstrates a form of adaptivity to the result of the first phase.

Key-words: differentiable optimization, multiobjective optimization, descent direction, Nash game

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Approches directe et adaptative en optimisation multiobjectif

Résumé : On formule de manière unifiée les principaux résultats théoriques obtenus par les auteurs dans le domaine de l'optimisation différentiable multobjectif, on discute des exemples illustratifs, et on présente brièvement les développements logiciels conduits à l'Inria s'y rapportant. Le développement se partage en deux parties liées. Dans la partie A, l'Algorithme de Descente à Multiples Gradients (MGDA), que l'on qualifie d'approche directe, est une construction générale de méthode de descente dans le contexte d'optimisation multiobjectif. L'algorithme fournit une technique pour déterminer des solutions Pareto optimales à des problèmes contraints comme une extension de la méthode classique du gradient. Dans la partie B, on pose une autre problématique, que l'on qualifie d'approche adaptative. Celle-ci se développe à la suite du choix d'une solution Pareto optimale vis à vis d'un ensemble de fonctions coûts principales soumises à des contraintes obtenue par application de MGDA, ou toute autre technique effective d'optimisation multiobjectif, possiblement une stratégie évolutionnaire. Cette deuxième phase de l'optimisation permet de construire un continuum de solutions proches pour lesquelles de nouvelles fonctions coûts, dites secondaires, sont réduites au prix d'une dégradation modérée de la condition de Pareto stationnarité des fonctions coûts principales. Ainsi, le processus d'optimisation complet démontre une forme d'adaptivité au résultat de la première phase.

Mots-clés : optimisation différentiable, optimisation multiobjectif, direction de descente, jeu de Nash

PART A: THE DIRECT APPROACH: MULTIPLE GRADIENT DESCENT ALGORITHM (MGDA)

Overview: The MGDA was first introduced in [4] as a fully general technique to identify a descent direction common to an arbitrary set of cost functions for which the parametric gradients at a given point are known. When the gradients are linearly independent the derivatives of the cost functions in the descent direction are equal. The construction has been described in general terms in [6]. Here the description is conducted in short for the purpose of Part B, and a numerical example in flow control is given as an illustration.

1 Construction and essential properties

Problematics and notations. Given m differentiable cost functions $\{f_j(\mathbf{x})\}$ ($j = 1, \dots, m$; $\mathbf{x} \in \Omega_{ad} \subseteq \mathsf{R}^n$), and their gradients at some admissible point $\mathbf{x}_0 \in \Omega_{ad}$,

$$\mathbf{g}_j = \nabla f_j(x_0), \tag{1}$$

find $\mathbf{d}^\star \in \mathsf{R}^n$ such that the directional derivatives <u>all</u> be positive:

$$D_{\mathbf{d}^\star} f_j(x_0) = (\mathbf{g}_j)^t \mathbf{d}^\star \geq 0 \quad (\forall j) \tag{2}$$

(superscript ^{t} for transposition).

If strict inequalities hold, the vector $(-\mathbf{d}^\star)$ provides a descent direction common to all cost functions. Such a direction exists if $\mathbf{x}_0$ is not (weakly-) Pareto-optimal.

MGDA Construction. Throughout, the Euclidean norm is considered in the finite-dimensional space. However it can be considered by reference to a general basis, other than canonical.

Definition 1 (Euclidean metrics)
Given an $n \times n$ positive-definite matrix $\mathbf{A}_n$, define the scalar product in R^n

$$(\mathbf{u} \in \mathsf{R}^n) \quad (\mathbf{v} \in \mathsf{R}^n) \quad \big(\mathbf{u}, \mathbf{v}\big) = \mathbf{u}^t \mathbf{A}_n \mathbf{v} \tag{3}$$

and associated Euclidean norm

$$(\mathbf{u} \in \mathsf{R}^n) \quad \|\mathbf{u}\| = \sqrt{\big(\mathbf{u}, \mathbf{u}\big)} = \sqrt{\mathbf{u}^t \mathbf{A}_n \mathbf{u}}. \tag{4}$$

Definition 2 (Convex hull of the family of gradients)

$$\overline{\mathbf{U}} = \left\{ \mathbf{u} \in \mathsf{R}^n \text{ such that: } \mathbf{u} = \sum_{j=1}^m \alpha_j \mathbf{g}_j,\ \alpha_j \geq 0\ (\forall j),\ \sum_{j=1}^m \alpha_j = 1 \right\} \tag{5}$$

Evidently the convex hull is a closed, bounded and convex set.

Proposition 1 (Element $\omega^\star$)
The convex hull $\overline{\mathbf{U}}$ admits a unique element $\boldsymbol{\omega}^\star$ of minimum Euclidean norm

$$\boldsymbol{\omega}^\star = \arg\min_{\mathbf{u} \in \overline{\mathbf{U}}} \|\mathbf{u}\|. \tag{6}$$

Proof: existence: $\overline{\mathbf{U}}$ is a finite-dimensional closed and bounded set; uniqueness: $\overline{\mathbf{U}}$ is convex. $\square$

Proposition 2 (Basic property of $\omega^\star$)
The element $\boldsymbol{\omega}^\star$ is such that:

$$\forall \mathbf{u} \in \overline{\mathbf{U}} : \quad \big(\mathbf{u}, \boldsymbol{\omega}^\star\big) \geq \|\boldsymbol{\omega}^\star\|^2 \tag{7}$$

and in particular:

$$\forall j : \quad \big(\mathbf{g}_j, \boldsymbol{\omega}^\star\big) = (\mathbf{g}_j)^t \mathbf{A}_n \boldsymbol{\omega}^\star \geq \|\boldsymbol{\omega}^\star\|^2 \tag{8}$$

and the element

$$\mathbf{d}^\star = \mathbf{A}_n \boldsymbol{\omega}^\star \tag{9}$$

is a common descent direction vector.

Proof: consider $\mathbf{u} \in \overline{\mathbf{U}}$, arbitrary, and let $\mathbf{v} = \mathbf{u} - \boldsymbol{\omega}^\star$. Since $\overline{\mathbf{U}}$ is convex, one has:

$$\forall \theta \in [0,1] : \ (1-\theta)\boldsymbol{\omega}^\star + \theta \mathbf{u} = \boldsymbol{\omega}^\star + \theta \mathbf{v} \in \overline{\mathbf{U}}. \tag{10}$$

Since $\boldsymbol{\omega}^\star$ is the minimum-norm element in $\overline{\mathbf{U}}$:

$$\|\boldsymbol{\omega}^\star + \theta \mathbf{v}\|^2 = \big(\boldsymbol{\omega}^\star + \theta \mathbf{v}, \boldsymbol{\omega}^\star + \theta \mathbf{v}\big) \geq \|\boldsymbol{\omega}^\star\|^2. \tag{11}$$

By developing the scalar product, one gets

$$\forall \theta \in [0,1] : \ 2\theta\big(\boldsymbol{\omega}^\star, \mathbf{v}\big) + \theta^2 \|\mathbf{v}\|^2 \geq 0 \tag{12}$$

and this requires that the coefficient of θ be positive:

$$\big(\boldsymbol{\omega}^\star, \mathbf{v}\big) = \big(\mathbf{u} - \boldsymbol{\omega}^\star, \boldsymbol{\omega}^\star\big) \geq 0. \tag{13}$$

□

Propositions 1-2 lead us to define the descent direction as follows:

Definition 3 (MGDA descent direction)

$$\mathbf{d}^\star = \mathbf{A}_n \boldsymbol{\omega}^\star, \quad \boldsymbol{\omega}^\star = \sum_{j=1}^{m} \boldsymbol{\alpha}_j^\star \mathbf{g}_j. \tag{14}$$

Note that the vector $\boldsymbol{\omega}^\star$ may be the minimum-norm element in the convex hull of a subfamily of the gradients, the remaining ones being redundant. In such a case, it is convenient to rearrange the ordering and split the indices in the two subfamilies

$$J^\star = \big\{\, j \in \{1,2,\dots,m\} \text{ such that } \alpha_j^\star \neq 0 \,\big\} \quad \text{(active)}, \tag{15}$$

$$J_0 = \big\{\, j \in \{1,2,\dots,m\} \text{ such that } \alpha_j^\star = 0 \,\big\} \quad \text{(redundant)}. \tag{16}$$

Then:

$$D_{\mathbf{d}^\star} f_j = (\mathbf{g}_j)^t \mathbf{d}^\star \begin{cases} = \sigma, & \forall j \in J^\star, \\ \geq \sigma, & \forall j \in J_0, \end{cases} \tag{17}$$

where:

$$\sigma = \|\boldsymbol{\omega}^\star\|^2. \tag{18}$$

MGDA iteration. We define the Multiple Gradient Descent Algorithm as the extension of the steepest descent method to the multi-objective context obtained by using the direction $(-\mathbf{d}^\star)$ as the search direction

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \epsilon_k \mathbf{d}_k^\star \tag{19}$$

where k is the iteration index, and ϵ_k the step-size.

The classical discussion on the step-size adjustment applies. A scaling was proposed in [5], but this discussion is eluded here for brevity.

Under standard assumptions, MGDA converges to "critical points" (or "weakly Pareto-optimal points") here referred to as "Pareto-stationary points" ($\boldsymbol{\omega}^\star = \mathbf{d}^\star = 0$). Hence, the limit points, since Pareto stationary, are candidates for Pareto optimality. To justify this statement more precisely, consider the following:

Proposition 3 (Pareto optimality, convexity and Pareto stationarity)
If the point $\mathbf{x}_0$ is Pareto optimal, and if all the cost functions are convex in some neighborhood of $\mathbf{x}_0$, then $\mathbf{x}_0$ is Pareto stationary ($\boldsymbol{\omega}^\star = \mathbf{d}^\star = 0$).

This result is known by many, but its proof is given here in the general case where m can be greater than n:

Proof: Let $\mathbf{g}_j = \nabla f_j(\mathbf{x}_0)$. Without loss of generality, suppose that $f_j(\mathbf{x}_0) = 0$ $(\forall j)$. Since $\mathbf{x}_0$ is Pareto optimal, any single cost function cannot be diminished (below 0) under the constraint of no-degradation of the others, a degradation corresponding to a strictly positive value. In particular, $\mathbf{x}_0$ is a solution to the problem of minimizing $f_m(\mathbf{x})$ under the constraint that the other cost functions are maintained ≤ 0, that is, the point $\mathbf{x}_0$ solves the problem:

$$\min_{\mathbf{x}} f_m(\mathbf{x}) \text{ subject to : } f_j(\mathbf{x}) \leq 0 \ (\forall j \leq m-1). \tag{20}$$

Let $\overline{\mathbf{U}}_{m-1}$ be the convex hull of the $m-1$ gradients $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{m-1}\}$ and

$$\boldsymbol{\omega}_{m-1}^\star = \arg \min_{\mathbf{g} \in \overline{\mathbf{U}}_{m-1}} \|\mathbf{g}\| . \tag{21}$$

The existence, uniqueness and the following property of this element have already been established (Propositions 1 and 2):

$$\left(\mathbf{g}_j, \boldsymbol{\omega}_{m-1}^\star\right) \geq \|\omega_{m-1}\|^2 \ \ (\forall j \leq m-1). \tag{22}$$

Two situations are then possible:

- either $\omega_{m-1} = 0$, and the Pareto-stationarity condition is satisfied at $\mathbf{x} = \mathbf{x}_0$ with $\alpha_m = 0$;
- or $\boldsymbol{\omega}_{m-1}^\star \neq 0$. Then let $\phi_j(\epsilon) = f_j(\mathbf{x}_0 - \epsilon\omega_{m-1})$ $(j = 1, \ldots, m-1)$ so that $\phi_j(0) = 0$ and $\phi_j'(0) = -\left(\mathbf{g}_j, \omega_{m-1}\right) \leq -\|\omega_{m-1}\|^2 < 0$ (strict inequality), and for sufficiently-small ϵ:

$$\phi_j(\epsilon) = f_j(\mathbf{x}_0 - \epsilon\omega_{m-1}) < 0 \qquad (\forall j \leq m-1). \tag{23}$$

This result confirms that for the constrained minimization problem (20), Slater's constraint-qualification condition [2] is satisfied. Hence, optimality requires the satisfaction of the Karush-Kuhn-Tucker (KKT) condition that is, the Lagrangian,

$$\mathsf{L} = f_m(\mathbf{x}) + \sum_{j=1}^{m-1} \lambda_j f_j(\mathbf{x}) \tag{24}$$

must be stationary, and this gives:

$$\mathbf{g}_m + \sum_{j=1}^{m-1} \lambda_j \mathbf{g}_j = 0 \tag{25}$$

in which $\lambda_j > 0$ $(\forall j \leq m-1)$ by saturation of the constraints $(f_j(\mathbf{x}_0) = 0)$ and sign convention. Finally, $\mu = 1 + \sum_{i=1}^{m-1} \lambda_i > 1$. Thus the above equation can be divided by $\mu \neq 0$ yielding the result. □

QP formulation and projection. Let $\mathbf{G}$ be the $n \times m$ Jacobian matrix

$$\mathbf{G} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{g}_1 & \mathbf{g}_2 & \dots & \mathbf{g}_m \\ \vdots & \vdots & & \vdots \end{pmatrix} \tag{26}$$

The element $\boldsymbol{\omega}^\star$ is by definition the solution of the following Quadratic Programming (QP) problem:

Minimum-norm element

$$\boldsymbol{\omega}^\star = \mathbf{G}\boldsymbol{\alpha}^\star \tag{27}$$

$$\boldsymbol{\alpha}^\star = \arg\min_{\boldsymbol{\alpha}} \tfrac{1}{2}\boldsymbol{\alpha}^t(\mathbf{G}^t\mathbf{A}_n\mathbf{G})\boldsymbol{\alpha} \tag{28}$$

$$\alpha_j \geq 0 \ (\forall j) \tag{29}$$

$$\mathbf{e}^t\boldsymbol{\alpha} = 1 \tag{30}$$

where $\mathbf{e}^t$ stands for the row vector $(1, 1, \dots, 1) \in \mathsf{R}^m$. If the positivity condition is abandoned, one obtains the projection onto a subspace containing $\overline{\mathbf{U}}$. This is a vector or affine subspace depending on the representation. In the affine representation it is a polytope of R^n.

Projected element

$$\boldsymbol{\omega}^\perp = \mathbf{G}\boldsymbol{\alpha}^\perp \tag{31}$$

$$\boldsymbol{\alpha}^\perp = \arg\min_{\boldsymbol{\alpha}} \tfrac{1}{2}\boldsymbol{\alpha}^t(\mathbf{G}^t\mathbf{A}_n\mathbf{G})\boldsymbol{\alpha} \tag{32}$$

$$\mathbf{e}^t\boldsymbol{\alpha}^\perp = 1 \tag{33}$$

Let us now examine the possibility to solve directly for the coefficient vector $\boldsymbol{\alpha}^\perp$. Define the Lagrangian

$$\mathsf{L} = \tfrac{1}{2}\boldsymbol{\alpha}^t(\mathbf{G}^t\mathbf{A}_n\mathbf{G})\boldsymbol{\alpha} + \lambda(\mathbf{e}^t\boldsymbol{\alpha} - 1) \tag{34}$$

and express the optimality conditions:

$$(\mathbf{G}^t\mathbf{A}_n\mathbf{G})\boldsymbol{\alpha}^\perp + \lambda\mathbf{e} = 0, \quad \mathbf{e}^t\boldsymbol{\alpha}^\perp = 1. \tag{35}$$

Hence the system to be solved is the following

$$\boldsymbol{\Gamma}\begin{pmatrix} \boldsymbol{\alpha}^\perp \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{36}$$

where:

$$\boldsymbol{\Gamma} = \begin{pmatrix} (\mathbf{G}^t\mathbf{A}_n\mathbf{G}) & \mathbf{e} \\ \mathbf{e}^t & 0 \end{pmatrix}. \tag{37}$$

Proposition 4
If the Jacobian matrix $\mathbf{G}$ is either of rank m, or of rank $m-1$ in a situation of Pareto-stationarity, the matrix $\boldsymbol{\Gamma}$ is invertible, and the coefficient vector $\boldsymbol{\alpha}^\perp$ of the projected element $\boldsymbol{\omega}^\perp$ is unique.

Proof: Matrix $\boldsymbol{\Gamma}$ is not invertible if and only if non trivial solutions to the homogeneous system

$$\boldsymbol{\Gamma}\begin{pmatrix} \boldsymbol{\beta} \\ \mu \end{pmatrix} = 0 \tag{38}$$

($\boldsymbol{\beta} \in \mathsf{R}^m,\ \mu \in \mathsf{R}$) exist, and this is equivalent to the following:

$$(\mathbf{G}^t\mathbf{A}_n\mathbf{G})\boldsymbol{\beta} + \mu\mathbf{e} = 0 \tag{39}$$

$$\mathbf{e}^t\boldsymbol{\beta} = 0. \tag{40}$$

Multiply (39) by $\boldsymbol{\beta}^t$ and use (40) to get $\|\mathbf{G}\boldsymbol{\beta}\|^2 = 0$, that is $\boldsymbol{\beta} \in \operatorname{Ker}\mathbf{G}$, and two cases are then possible:

- either $\mathbf{G}$ is of rank m and $\operatorname{Ker}\mathbf{G} = \{0\}$, and this implies successively: $\boldsymbol{\beta} = 0$ and $\mu = 0$, and only the trivial solution is found;
- or $\mathbf{G}$ is of rank $m-1$ and $\dim \operatorname{Ker} \mathbf{G} = 1$.

By assumption, the latter case occurs in a situation of Pareto stationarity for which there exists a vector $\boldsymbol{\alpha}_0$ such that $\mathbf{G}\boldsymbol{\alpha}_0 = 0$ where the components of α_0 are positive and of sum equal to 1. Then $\operatorname{Ker} \mathbf{G} = \{k\boldsymbol{\alpha}_0\}$ ($k \in \mathsf{R}$). Then successively: $\boldsymbol{\beta} = k\boldsymbol{\alpha}_0$, $k = 0$ (by substitution in (40)) (and since $\mathbf{e}^t\boldsymbol{\alpha}_0 = 1$), $\boldsymbol{\beta} = 0$, $\mu = 0$, and again only the trivial solution is found.

In both cases, $\boldsymbol{\Gamma}$ is invertible, and $\boldsymbol{\alpha}^{\perp}$ is unique. □

The interest for considering $\boldsymbol{\omega}^{\perp}$ resides in the fact that if $\boldsymbol{\alpha}^{\perp}$ can be computed easily, and if it has positive components, $\boldsymbol{\omega}^{\perp} \in \overline{\mathbf{U}}$ and $\boldsymbol{\omega}^{\star} = \boldsymbol{\omega}^{\perp}$. This observation led us to define the second method for computing $\boldsymbol{\omega}^{\star}$ described below.

First method for computing $\boldsymbol{\omega}^{\star}$. Our most general method has been implemented on the platform `http://mgda.inria.fr` to which the reader can refer for a detailed presentation of the software and utilization. The software applies to cases for which $m > n$ as well as those for which $m \leq n$, more typical of multi-disciplinary optimization. In short, the algorithm consists of the following:

- Gram-Schmidt orthogonalization process applied to the set of gradients $\{\mathbf{g}_j\}$ made special by
  - a hierarchical principle for pivoting
  - a particular normalization for scaling.

  Certain precise conclusions on directional derivatives can be drawn at completion of the process [6].
- Whenever full conclusion cannot be achieved by the above process, which usually occurs when $m > n$, the QP problem is reformulated in a favorable basis (avoiding ill-conditioning; $\mathbf{A}_n \neq \mathbf{I}_n$) and solved by QPGEN2.F (public-domain procedure).

Illustration of the method in the optimization of a system of jets. For purpose of illustration of the method, a numerical result fully described in [9] is outlined here.

In this numerical experiment, the two-dimensional time-dependent viscous flow over a flat plate, governed by the full Navier-Stokes equations in the compressible and laminar regime, was computed. The device includes three pulsating jets acting as time-periodic local boundary conditions. As a result, the flow was periodic, but not sinusoidal, and depended parametrically on the prescribed 6-component vector

$$\mathbf{x} = \{A_1, A_2, A_3, \varphi_1, \varphi_2, \varphi_3\}. \tag{41}$$

of amplitudes and phases of the jets. For a given $\mathbf{x}$, the integration of the flow over a dozen periods, typically, permitted to achieve a quasi-periodic regime, yielding the drag force by spatial integration along the plate. This integration was conducted at every time-step of the time-stepping integration and this provided the graph of the unsteady drag over a time period.

The sensitivity equations w.r.t. each component of $\mathbf{x}$ were also integrated simultaneously with the flow equations, and this permitted to calculate the gradient of drag w.r.t. $\mathbf{x}$.

In this way, we obtained for the 800 time-steps of the time integration, not only the corresponding values of time-dependent drag, but also of its gradient w.r.t. $\mathbf{x}$. In order to reduce the problem dimension, each element of this large dataset was averaged over 40 time-steps, yielding 20 averaged drag values, and 20 averaged gradient vectors of dimension 6. These gradients were used as an input to the MGDA optimization process to identify a common descent direction, then used to update the jet system, and the whole process was repeated 5 times to optimize $\mathbf{x}$ and globally reduce drag over the entire time period.

The drag force as a function of time is represented over a period on Figure 1 at different stages of the optimization:

- the horizontal green dashed line is the baseline; it gives the value of drag when the jets are not operated;
- the blue curve corresponds to drag in the somewhat arbitrary initial setting; evidently, the introduction of the jets does reduce the average drag, but not at all times;
- the black curves represent drag at different stages of the optimization by MGDA;
- the red line represents drag after the MGDA process has converged.

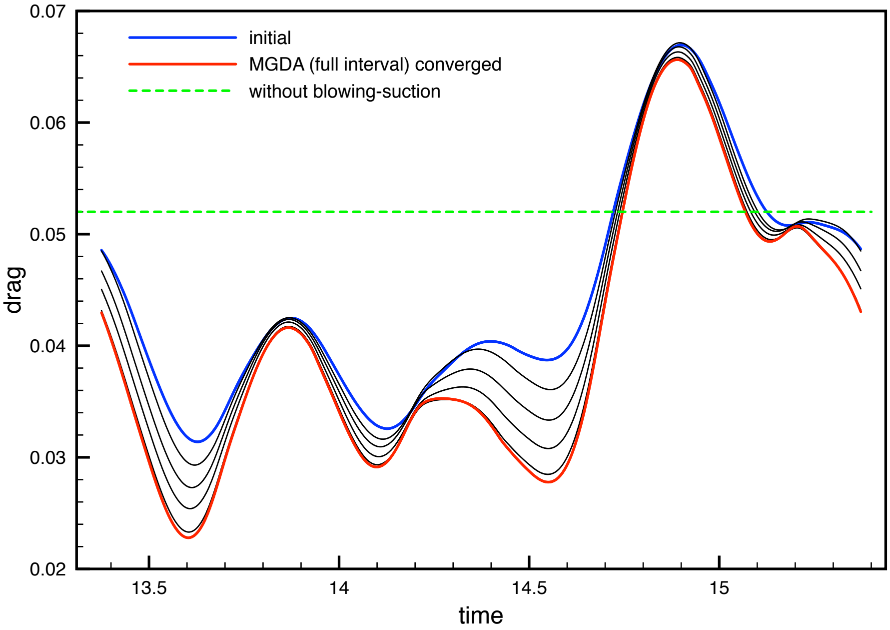

Figure 1: Design optimization of a system of pulsating jets over a flat plate. Drag force over a time period, at different stages of the optimization process The blue and red curves correspond to the initial and final settings of the jets.

The figure confirms the ability of MGDA to reduce the 20 cost functions simultaneously, and in fact, the 800 drag values as verified a posteriori. Of course, the reduction of the average drag is inferior with MGDA than it would be by minimizing this functional alone, but it is effective at all times. By its potential to act directly on the time-dependent function, MGDA offers the practitioner numerous possible optimization strategies of such devices. In particular, in [9], a similar experiment was conducted to concentrate the optimization on the sole 40% tail portion of the period, yielding a very different setting of the jets.

In conclusion, MGDA offers the potential, in a multi-point problem, to be active at all points, and not simply in the mean.

Second method for computing $\boldsymbol{\omega}^\star$. This alternative applies to cases of linearly-independent gradients. It has been developed in cooperation with L. Hascoët, T. Kloczko and L. Monasse (Inria). It is a recursive algorithm alternating projections to determine a potential $\boldsymbol{\omega}^\perp$ and tests of admissibility ($\boldsymbol{\alpha}^\perp \in \overline{\mathbf{U}}$?). The algorithm is very efficient but has only been tested for relatively small dimension $m \le n$. Here the setting has the following characteristics:

- The unknown $\boldsymbol{\alpha}^\star = (\alpha_1, \dots, \alpha_m)^\star$ is unique.

- The QP formulation is considered in the canonical basis ($\mathbf{A}_n = \mathbf{I}_n;\ \mathbf{d}^\star = \boldsymbol{\omega}^\star$).
- In short: in the affine representation, the origin O is projected onto the polytope associated to $\overline{\mathbf{U}}$ and the vector $\omega^\perp$ is defined; if admissible ($\omega^\perp \in \overline{\mathbf{U}}$), $\boldsymbol{\omega}^\star = \omega^\perp$ and the solution is found; otherwise, $\boldsymbol{\omega}^\star$ belongs to the boundary of a polytope of R^n having m vertices. In large dimension, this boundary can be made of elements of many types: points, edges, facets, ... convex subsets of subspaces of dimension $\leq m-1$. Each type is associated with a subset of coefficients $\{\alpha_j\}$ equal to 0. Then one is led to explore (at most) 2^m branches of a tree associated with this occurrence: from every non-admissible projected point originate two new sub-branches associated with one more coefficient set to 0 until an edge is reached (2 nonzero coefficients) for which the solution is known. One retains the best solution among endpoints of all branches.

The exploration of the polytope boundary is illustrated in Figures 2 and 3 for the cases of two and three gradients. In dimension 3, the determination of $\boldsymbol{\omega}^\star$ from $\boldsymbol{\omega}^\perp$ is already tricky: it may not result from the projection onto the closest edge.

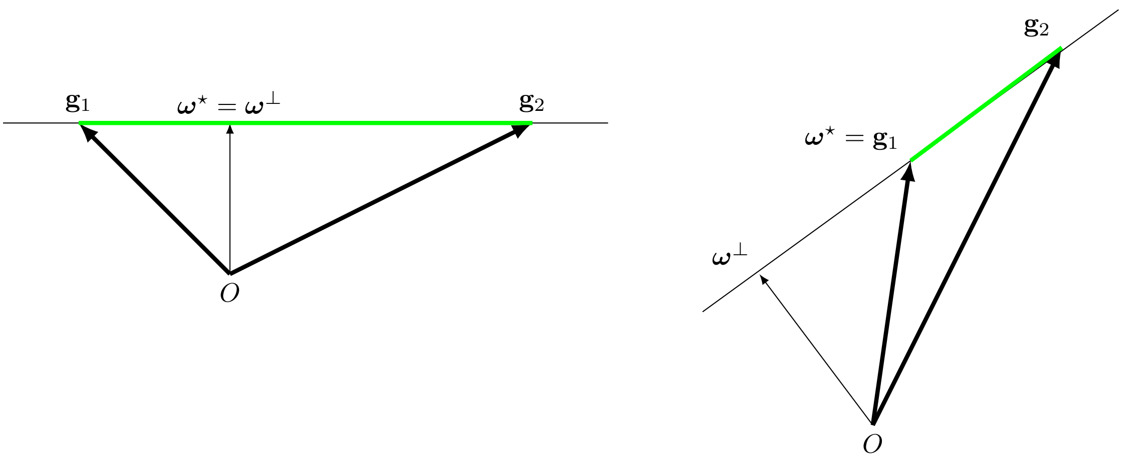

Figure 2: Case of two gradient vectors: $\overline{\mathbf{U}}$ is made of all the vectors of origin O that point on the edge limited by $\{\mathbf{g}_1, \mathbf{g}_2\}$. In the first case (left), $\boldsymbol{\omega}^\star = \boldsymbol{\omega}^\perp$ results from the orthogonal projection of O onto the edge; in the second case (right), $\boldsymbol{\omega}^\star$ is the gradient closer to this projection $\boldsymbol{\omega}^\perp$.

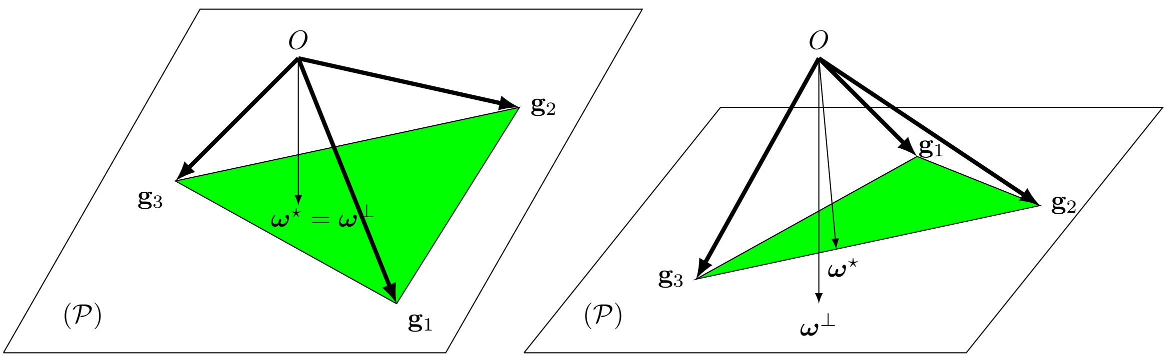

Figure 3: Case of three gradient vectors: $\overline{\mathbf{U}}$ is the set of vectors of origin O that point on the triangle limited by $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$. On the left, the case where the orthogonal projection of O onto the plane of the triangle belongs to the triangle; then $\boldsymbol{\omega}^\star = \boldsymbol{\omega}^\perp$. On the right, the inverse case in which $\boldsymbol{\omega}^\star$ is found by exploring the boundary of the triangle, made of three edges corresponding to $\alpha_1 = 0$, $\alpha_2 = 0$ or $\alpha_3 = 0$.

The recursive tree exploration is sketched on Figure 4.

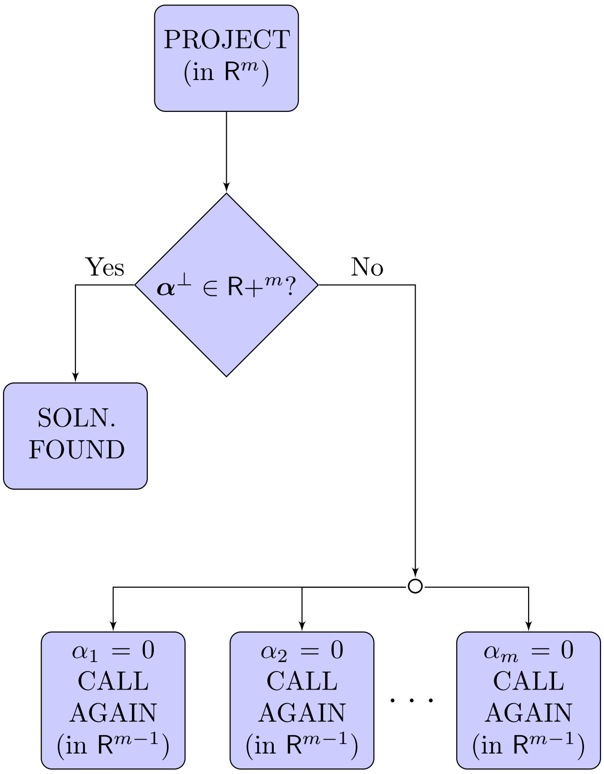

Figure 4: Sketch of recursive tree exploration. Note that when a branch is completed, the result serves to initiate or upgrade the best result found.

2 Conclusions and perpectives for MGDA

In summary, MGDA relies on a simple and general property of convex geometry used to formulate a variational principle that defines a descent direction common to all cost functions, as the solution of a QP-problem. The method is effective in all situations of differential optimization except when the evaluation point is Pareto stationary.

It should be emphasized that in the construction of the descent direction, no assumption is made on the way the number m of cost functions and the dimension n of the design space compare. Thus it applies to multi-objective optimization problems in which

- $m \leq n$, typical of Multi-Disciplinary Optimization,
- as well as, $m > n$ and even $m \gg n$ occurring in certain multi-point optimization problems, as illustrated in the optimization of a flow-control device governed by the time-dependent compressible Navier-Stokes equations.

Two techniques have been implemented to compute the descent direction. The first, and most general, relies on a Gram-Schmidt orthogonalization process that incorporates special options for pivoting and normalization, and yields certain precise bounds at completion of the process. When the numerical solution of the QP-problem reveals necessary, it is performed after a change of basis has been realized. This improves notably the problem conditioning when the dimensions are large. This procedure is made available on the Inria MGDA software platform `http://mgda.inria.fr`. In the second procedure, the boundary of a polytope in dimension n is explored by a recursive algorithm.The latter procedure has been tested in many cases; however in these cases, the dimensions (m, n) were relatively small.

Finally, MGDA has been extended to stochastic formulations by works conducted at ONERA jointly with F. Poirion (research engineer) and Q. Mercier (former doctoral student) [17]. These works have led to contributions in

– non smooth multi-objective stochastic optimization [17] [19]

– non convex multi-objective optimization under uncertainties with application to aero-elasticity with material variability [20].

Part B: The adaptive approach: multi-objective optimization prioritized by by Nash games

Overview: One considers a multi-objective differential optimization problem involving two sets of cost functions: *(i)* a primary set of prioritized cost functions $\{f_j(\mathbf{x})\}$ $(j = 1, \dots, m)$, and *(ii)* a secondary set of cost functions $\{f_j(\mathbf{x})\}$ $(j = m+1, \dots, M)$ of lesser importance, where $\mathbf{x} \in \Omega_a$ $(\Omega_a \subseteq \mathsf{R}^n$: admissible domain). The cost functions are smooth, say of class $C^2(\Omega_a)$, and the problem is subject to the equality constraint $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), \dots, c_K(\mathbf{x})) = 0$ $(K \leq n-2)$. In a first phase of optimization only the prioritized cost functions are minimized. It is assumed that this phase is conducted successfully using some effective multi-objective optimization method and permits to identify a valid Pareto-optimal solution $\mathbf{x}_A^\star$. At this point, a representative function of the subset of primary cost functions, $f_A(\mathbf{x})$, is defined specifically. If $m = 1$, $f_A = f_1$. The question of convexity is examined closely. The function f_A is augmented by an adequate convexity-fix term, and this results in the definition of a primary steering function $f_A^+(\mathbf{x})$ locally minimum and strictly-convex in the neighborhood of $\mathbf{x}_A^\star$.

The Hessian matrix $\nabla^2 f_A^+(\mathbf{x}_A^\star)$ and the constraint gradients $\{\nabla c_k(\mathbf{x}_A^\star)\}$, assumed to be linearly-independent, are computed and used to decompose R^n into two supplementary subspaces U and V ("territory splitting").

Elements of the Multiple-Gradient Descent Algorithm are used to define precisely a secondary steering function $f_B(\mathbf{x})$ representing globally the secondary cost functions. If $M = m+1$, $f_B = f_M$.

A Nash game formulation is constructed in which two virtual players A and B negotiate an equilibrium between f_A^+ and f_B using strategies $\mathbf{u} \in U$ and $\mathbf{v} \in V$ respectively. Partial gradients of the secondary cost functions w.r.t a newly-defined variable $\mathbf{w}$ are calculated. Under the condition that these partial gradients are not in a configuration of Pareto-stationarity, the formulation is proved to provide a continuum of Nash equilibria parameterized by a small parameter ε and such that: *(i)* for $\varepsilon = 0$, the Nash equilibrium point is $\mathbf{x}_A^\star$; *(ii)* as ε increases, the stationarity of f_A and f_A^+, equivalent to the Pareto-stationarity of the prioritized cost functions, is preserved to second-order in ε, whereas f_B, and all secondary cost functions with it, decrease (at least) linearly with an initial derivative at most equal to $-\sigma_B$, where σ_B is a strictly-positive constant known a priori.

The method is illustrated by the numerical treatment of analytical test-cases, as well as an aircraft design problem involving 15 variables in which mass, range, approach speed and take-off distance are optimized subject to the classical laws of flight mechanics.

3 Introduction and hypotheses

In design optimization the practitioner is often led to account for a large variety of cost functions of greater or lesser importance to the system under consideration. Hence it may be opportune to introduce a certain prioritization among cost functions. Some of these cost functions are sometimes treated alternately as constraints. One would expect that in the initial phase of an extensive campaign of optimization of a complex system, only a "primary cost function" $f_A(\mathbf{x}) = f_1(\mathbf{x})$, or possibly a set of "primary cost functions" $\{f_j(\mathbf{x})\}$ $(j = 1, \dots, m)$, essential to the well-functioning and performance of the system, is subject to the minimization w.r.t the design vector $\mathbf{x} = (x_1, \dots, x_n) \in \Omega_a$ ($\Omega_a \subseteq \mathsf{R}^n$: admissible set). This minimization is subject to a set of constraints. Here we only consider smooth equality constraints,

$$\mathbf{c} = \mathbf{c}(\mathbf{x}) = \big(c_1(\mathbf{x}), \dots, c_K(\mathbf{x})\big) = 0 \tag{42}$$

where $K < n$, and smooth cost functions (say $C^2(\Omega_a)$) of the variable $\mathbf{x}$.

We assume that a point $\mathbf{x}_A^\star$ of Pareto-optimality of the prioritized cost functions alone, subject to the constraints $\mathbf{c}(\mathbf{x}) = 0$, is known. From the computational viewpoint, such a point $\mathbf{x}_A^\star$ can be calculated by application of the constrained Multiple-Gradient Descent Algorithm of Part A initialized at some appropriate starting point, or some other effective multi-objective optimization method; perhaps an evolutionary algorithm. In general, the limiting point depends on the starting point, or the starting set of points. However, we make the somewhat subjective assumption, that $\mathbf{x}_A^\star$ is indeed a valid operational point w.r.t the prioritized cost functions considered alone, subject to the constraints, and our objective is to reduce the secondary cost functions in the neighborhood of this point, while maintaining the Pareto-stationarity condition at best.

Note that the secondary cost functions are sometimes known from start, but inversely, can also be revealed by the a posteriori detailed analysis of the design point $\mathbf{x}_A^\star$. We expect the latter case to occur in the optimization of a complex multidisciplinary system.

In this part specifically, we have adopted the option of considering logarithmic rather raw gradients for cost functions, aiming to reduce them in comparable proportions. To permit this, the cost functions are all assumed to be uniformly strictly positive. This may require a reformulation of the initial setting by the substitution of newly-defined cost functions in place of the former ones, still respecting their sense of variation. This can easily be achieved by application, for example, of an exponential transform with an adequate scaling of the exponent. Note that such a transform modifies the norm of the gradients and Hessians. Hence it can, and should be done in a way that improves the problem numerical conditioning.

Thus, at $\mathbf{x}_A^\star$, the following stationarity condition is satisfied:

$$\sum_{j=1}^{m} \alpha_j^\star \mathbf{g}_j^\star = 0, \quad \mathbf{c}^\star = 0, \tag{43}$$

where:

- the superscript * on any symbol indicates an evaluation at $\mathbf{x} = \mathbf{x}_A^\star$;
- the logarithmic gradients have been projected onto the subspace tangent to the constraint gradients:

$$\mathbf{g}_j = \frac{\mathbf{P}\nabla f_j^\star}{f_j^\star} \quad (j = 1, \dots, m) \tag{44}$$

  in which ∇ is the symbol for the gradient-vector w.r.t $\mathbf{x}$;
- $\{\alpha_j^\star\}$ $(j = 1, \dots, m)$ is a known set of convex coefficients: $\alpha_j^\star \geq 0$, $\forall j$; $\sum_{j=1}^m \alpha_j^\star = 1$;
- the constraint gradients $\{\nabla c_k^\star\}$ $(k = 1, \dots, K)$ are assumed to be linearly independent, a standard hypothesis for "constraint qualification" [13];
- $\mathbf{P}$ is the projection matrix onto the subspace tangent to the constraint manifold which is, orthogonal to all constraint gradients $\{\nabla c_k^\star\}$.

Note that (44) thereafter replaces the definition of gradients utilized in Part A, (1).

The Gram-Schmidt orthogonalization process is applied to the constraint gradients, technically by the QR factorization. This yields a set of 2×2-orthonormal vectors, $\{\mathbf{q}^k\}$ $(k = 1, \dots, K)$, spanning the same subspace. Then

$$\mathbf{P} = \mathbf{I}_n - \sum_{k=1}^{K} \left[\mathbf{q}^k\right] \left[\mathbf{q}^k\right]^t \tag{45}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\left[\mathbf{q}^k\right]$ stands for the column-vector made of the components of $\mathbf{q}^k$ in the canonical basis, and the superscript ^{t} indicates transposition. The Pareto-stationarity condition in (43) can be written equivalently in Lagrangian form as follows:

$$\nabla f_A^\star + \sum_{k=1}^{K} \lambda_k \nabla c_k^\star = \nabla f_A^\star + \sum_{k=1}^{K} \underline{\lambda}_k \mathbf{q}^k = 0 \tag{46}$$

where the $\{\lambda_k\}$ are Lagrange multipliers and the following definition is made

$$f_A(\mathbf{x}) = \sum_{j=1}^{m} \alpha_j^\star \frac{f_j(\mathbf{x})}{f_j^\star}. \tag{47}$$

These definitions being made, the agglomerated cost function $f_A(\mathbf{x})$ is viewed thereafter as the representative of the whole set of primary cost functions.

Convexity fix. The agglomerated cost function $f_A(\mathbf{x})$ may not be convex at $\mathbf{x}_A^\star$. In such a case, one simply substitute to it, in the entire present development, the following augmented cost function:

$$f_A^+(\mathbf{x}) = f_A(\mathbf{x}) + \frac{c}{2}\left(\mathbf{x} - \mathbf{x}_A^\star, \mathbf{x} - \mathbf{x}_A^\star\right) \tag{48}$$

where the "convexity-fix constant" c is chosen large enough for the Hessian matrix

$$\mathbf{H}_A^{+\star} = \nabla^2 f_A^{+\star} \tag{49}$$

to be positive-definite. From the standpoint of implementation, this modification does not introduce additional numerical complexity (see 4.2). Evidently, the augmented cost function admits the same minimum $f_A^\star$ as the original one, reached at the same point $\mathbf{x}_A^\star$, but is there locally strictly-convex.

We further make the natural hypothesis that the Lagrangian

$$\mathsf{L} = f_A^+(\mathbf{x}) + \sum_{k=1}^{K} \lambda_k c_k(\mathbf{x}) \tag{50}$$

is convex at $\mathbf{x} = \mathbf{x}_A^\star$. This condition is met either because all constraint functions $\{c_k(\mathbf{x})\}$ are locally convex, or by further increase of the convexity-fix constant c. For a practical procedure to set this constant, see [8].

Hence, the point $\mathbf{x}_A^\star$ is a point of weak-Pareto optimality of the set of primary cost functions, and a point of local minimum of the cost function $f_A^+(\mathbf{x})$ representative of the prioritized set of cost functions under the constraint $\mathbf{c}(\mathbf{x}) = 0$.

Position of the problem. We now return to the engineering perspective. Optimal or "extremal" solutions w.r.t. a given discipline, by nature, often correspond to designs judged extreme, that is, unacceptable by the specialists of other disciplines. For example, in the aeronautical field, the optimum-shape design in the aerodynamic sense can be a very cambered geometry locally exhibiting small radii of curvature, and such a geometry is likely to be fragile, and rejected from the structural-analysis standpoint.

Hence, once the point $\mathbf{x}_A^\star$ is identified, the following question may be raised: how can $\mathbf{x}$ be modified from $\mathbf{x}_A^\star$ to improve, that is reduce, a set of smooth "secondary cost functions" $\{f_j(\mathbf{x})\}$ $(j = m+1, \dots, M)$ while best preserving the local constrained minimum of the cost function f_A^+?

We first address this problem in Section 4 in the two-discipline case ($m = 1$, $M = 2$) for which the functions $f_A = f_1/f_1^\star$ and $f_B = f_2/f_2^\star$ are problem specifications and require no special construction. The working space R^n is decomposed into two supplementary subspaces:

$$\mathsf{R}^n = U \oplus V. \tag{51}$$

The decomposition is referred to as a territory splitting. It applies to two virtual players A and B engaged in a specialized Nash game, and whose strategies are respectively $\mathbf{u} \in U$ and $\mathbf{v} \in V$. A parameterized continuum of Nash equilibria $\{\bar{\mathbf{x}}_\varepsilon\}$ ($\varepsilon \geq 0$), originating at $\varepsilon = 0$ from the constrained optimum of $f_A^+(\mathbf{x})$ ($\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$) is identified to be a solution. Both analytical test-cases, and a test-case in optimum-shape design in aerodynamics are used to illustrate the corresponding computational method.

Then, a generalization to the case of several secondary disciplines ($M > m+1$), actually in arbitrary number, is developed using again MGDA-related elements in the construction. The condition under which the Nash game results in an effective reduction of the secondary cost functions while maintaining the primary representative cost function quasi-optimality is established. The theoretical developments are illustrated by analytical test-cases as well as a test-case of optimum design in aeronautics.

4 Nash-game with territory splitting for two-discipline optimization

4.1 General framework

A special formulation of Nash game has been developed for two-discipline problems in which one discipline, A, is considered "primary", for being of preponderant importance or fragile, and the other, B, "secondary" [10]. It is assumed that the two disciplines share the same finite set of optimization variables, $\mathbf{x}$ ($\mathbf{x} \in \Omega_a$), that determine the cost functions f_A and f_B respectively, in a "parameter-optimization" framework. It is now assumed that the number K of scalar constraints is at most equal to $n-2$, and usually much less.

In computationally-demanding applications such as those of PDE-constrained optimization problems, the calculation of first-order derivatives, and *a fortiori* of second-order derivatives by adjoint formulations, although more commonly employed today, can still represent a complex development task. However, one can alternately resort to approximate derivatives by finite-differences or through meta-modelling if more convenient [15]. Thus, in addition to the K constraint gradients, we assume that the $n \times n$ Hessian matrix $\mathbf{H}_A^{+\star}$ of (49) is also known, exactly or sufficiently accurately by numerical approximation.

By applying the Gram-Schmidt orthogonalization process to the constraint gradients, the projection matrix $\mathbf{P}$ onto the the manifold orthogonal to these vectors has been identified, and one computes the following $n \times n$ reduced Hessian matrices

$$\mathbf{H}'_A = \mathbf{P}\mathbf{H}_A^\star\mathbf{P}, \quad \mathbf{H}_A^{+'} = \mathbf{P}\mathbf{H}_A^{+\star}\mathbf{P}, \quad \mathbf{H}_A^{+\star} = \mathbf{H}_A^\star + c\mathbf{I}_n. \tag{52}$$

Matrix $\mathbf{H}_A^{+'}$ is positive semi-definite, of rank $n-K$ exactly since matrix $\mathbf{H}_A^{+\star}$ is strictly-positive definite after the convexity-fix is made. These definitions also extend to the simpler unconstrained case ($K=0$) for which, conventionally, $\mathbf{P} = \mathbf{I}_n$ and $\mathbf{H}'_A = \mathbf{H}_A^\star$.

By diagonalization:

$$\mathbf{H}_A^{+'} = \mathbf{\Omega}\mathcal{H}\mathbf{\Omega}^t \tag{53}$$

where the matrix $\mathbf{\Omega}$ is orthogonal ($\mathbf{\Omega}^t\mathbf{\Omega} = \mathbf{I}_n$), and the matrix $\mathcal{H} = \mathbf{Diag}(h'_k)$ diagonal.

The column-vectors of matrix $\mathbf{\Omega}$ are the eigenvectors of matrix $\mathbf{H}'_A$. The eigenmodes are arranged in a special order. Among the eigenvectors, one finds those spanning the null space of $\mathbf{P}$, that is, the subspace spanned by the constraint gradients: these are precisely $\{\mathbf{q}^1, \ldots, \mathbf{q}^K\}$. They are placed first. Then, the remaining ones, denoted by extension $\{\mathbf{q}^k\}$ ($k = K+1, \ldots, n$), are placed in descending order of the eigenvalue (or "sensitivity") h'_k:

$$h'_1 = \cdots = h'_K = 0; \; h'_{K+1} \geq h_{K+2} \geq \cdots \geq h'_n > 0. \tag{54}$$

Then the following change of variables (or "territory splitting") is introduced:

$$\mathbf{x} = \mathbf{x}_A^\star + \mathbf{\Omega}\begin{pmatrix} \mathbf{g} \\ \mathbf{v} \end{pmatrix} := \mathbf{X}(\mathbf{u}, \mathbf{v}) \tag{55}$$

where $\mathbf{u} \in \mathsf{R}^{n-p}$ and $\mathbf{v} \in \mathsf{R}^{p}$ and the integer p, dimension of subspace V, is chosen such that $1 \le p < n - K$, so that $n - p > K$. Note that since the matrix $\mathbf{\Omega}$ is orthogonal, hence invertible, the mapping is one-to-one; additionally it is smooth, since linear. Therefore, the mapping is a valid change of variables for differentiable optimization in R^n.

We now make the following:

Hypothesis 1
Bounds $B_\mathbf{u}$ and $B_\mathbf{v}$ are assumed to be such that under the conditions

$$\|\mathbf{u}\| \le B_\mathbf{u}, \quad \|\mathbf{v}\| \le B_\mathbf{v}, \tag{56}$$

the function $f_A^+\left(\mathbf{X}(\mathbf{u},\mathbf{v})\right)$ admits a unique constrained minimum attained at $\mathbf{u} = 0$ and $\mathbf{v} = 0$, that is $\mathbf{x} = \mathbf{X}(0,0) = \mathbf{x}_A^\star$.

Consistently, we redefine the admissible domain to be:

$$\Omega'_a = \{\mathbf{x} = \mathbf{X}(\mathbf{u},\mathbf{v}) \text{ subject to (56)}\} \tag{57}$$

In this way, on one hand, for fixed $\mathbf{v} = 0$, the range of $\mathbf{X}(\mathbf{u},\mathbf{v})$ as $\mathbf{u}$ varies contains the subspace $\mathcal{T}$ tangent to the constraint manifold ($\mathbf{c} = 0$), and more; on the other hand, for fixed $\mathbf{u} = 0$, the range of $\mathbf{X}$ as $\mathbf{v}$ varies is at least one-dimensional.

Lastly, define a continuation parameter ε ($0 \le \varepsilon \le 1$), an under-relaxation parameter θ ($0 < \theta \le 1$), and the auxiliary cost function:

$$f_{AB} = f_A^+ + \varepsilon\left(\theta f_B - f_A^+\right). \tag{58}$$

Note that for $\varepsilon = 0$, $f_{AB} = f_A^+$.

Nash game. The sub-vectors $\mathbf{u}$ and $\mathbf{v}$ are chosen to be the "strategies" of two virtual players A and B engaged in a Nash game in which:

- <u>Strategy A :</u> *Player A* attempts to minimize $f_A^+\big(\mathbf{X}(\mathbf{u},\mathbf{v})\big)$ by the strategy $\mathbf{u}$, subject to the equality constraint $\mathbf{c}\big(\mathbf{X}(\mathbf{u},\mathbf{v})\big) = 0$, and by accounting for *Player B*'s fixed strategy $\mathbf{v}$,

whereas:

- <u>Strategy B :</u> *Player B* attempts to minimize $f_{AB}(\mathbf{X}(\mathbf{u},\mathbf{v}))$ by the strategy $\mathbf{v}$, subject to <u>no constraints</u>, but by accounting for *Player A*'s fixed strategy $\mathbf{u}$.

For examples of successful Nash games of this type in computational aerodynamic design optimization including implementation, see e.g. [10] or [21].

In this formulation, for a given ε, the vector $\bar{\mathbf{x}}_\varepsilon = \mathbf{X}(\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon)$ is a Nash equilibrium point, if and only if:

$$\bar{\mathbf{u}}_\varepsilon = \arg\min_{\mathbf{u}} f_A^+\Big(\mathbf{X}(\mathbf{u},\mathbf{v})\Big) \quad \text{subject to: } \mathbf{c}\Big(\mathbf{X}(\mathbf{u},\mathbf{v})\Big) = 0 \quad (\text{for fixed } \mathbf{v} = \bar{\mathbf{v}}_\varepsilon), \tag{59}$$

and:

$$\bar{\mathbf{v}}_\varepsilon = \arg\min_{\mathbf{v}} f_{AB}(\mathbf{X}(\mathbf{u},\mathbf{v})) \quad (\text{for fixed } \mathbf{u} = \bar{\mathbf{u}}_\varepsilon) \tag{60}$$

where in both equations $(\mathbf{u},\mathbf{v}) \in \Omega'_a$. This formulation differs slightly from the classical concept of Nash games, since it is not completely symmetrical: only *Player A* is directly subject to the specified equality constraint $\mathbf{c} = 0$. More importantly, the formulation raises three fundamental questions: *(i)* Does the Nash equilibrium exist? *(ii)* Can we compute the equilibrium solution by a simple "variable-passing" iteration (as in [21])? *(iii)* Is the equilibrium solution physically acceptable? What follows permits us to bring positive answers to all three questions.

Proposition 5 (Consistency)
At $\mathbf{x} = \mathbf{x}_A^\star$, the primary cost function $f_A^+(\mathbf{x})$ is convex, and the constraint gradients $\{\nabla c_k^\star\}$ ($1 \leq k \leq K$) are assumed to be linearly independent. Then, the solution to the primary optimization problem, $\mathbf{x}_A^\star$, is a Nash equilibrium for $\varepsilon = 0$:

$$\mathbf{x}_A^\star = \mathbf{X}(0,0) = \bar{\mathbf{x}}_0 \tag{61}$$

in the sense of (59)-(60) (equivalently: $\bar{\mathbf{u}}_0 = 0 \in \mathsf{R}^{n-p}$ and $\bar{\mathbf{v}}_0 = 0 \in \mathsf{R}^p$).

Proof: Note that the primary cost function f_A^+ is involved in three distinct optimization problems:

Pb_1 : "single-discipline constrained-minimization of f_A^+ in the whole admissible domain Ω'_a":

$$\min_{(\mathbf{u},\mathbf{v})\in\Omega'_a} f_A^+\big(\mathbf{X}(\mathbf{u},\mathbf{v})\big) \quad \text{subject to } \mathbf{c}\big(\mathbf{X}(\mathbf{u},\mathbf{v})\big) = 0; \tag{62}$$

Pb_2 : "single-discipline constrained-minimization of f_A^+ in the subset of the admissible domain Ω'_a lying in the affine subspace generated by *Player A*'s strategy $\mathbf{u}$, subject to *Player B*'s strategy (at $\varepsilon = 0$) $\mathbf{v} = \bar{\mathbf{v}}_0 = 0$":

$$\min_{\mathbf{u}} f_A^+\big(\mathbf{X}(\mathbf{u},0)\big) \quad \text{subject to } \mathbf{c}\big(\mathbf{X}(\mathbf{u},0)\big) = 0 \text{ and } \|\mathbf{u}\| \leq B_{\mathbf{u}}; \tag{63}$$

Pb_3 : "single-discipline unconstrained-minimization of $f_{AB} = f_A^+$ (for $\varepsilon = 0$), in the subset of the admissible domain Ω'_a lying in the affine subspace generated by *Player B*'s strategy $\mathbf{v}$, subject to *Player A*'s strategy (at $\varepsilon = 0$) $\mathbf{u} = \bar{\mathbf{u}}_0 = 0$":

$$\min_{\mathbf{v}} f_A^+(\mathbf{X}(0,\mathbf{v})) \quad (\|\mathbf{v}\| \leq B_{\mathbf{v}}). \tag{64}$$

We aim to prove that all three optimization problems, although distinct by their respective admissible domains, share the same solution $\bar{\mathbf{u}} = \bar{\mathbf{u}}_0 = 0 \in \mathsf{R}^{n-p}$, $\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 = 0 \in \mathsf{R}^p$, corresponding to $\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$. The reasons are the following:

Problem Pb_1 is the original first-phase optimization problem reformulated in terms of the variable $(\mathbf{u},\mathbf{v})$. It admits the solution $(\mathbf{u},\mathbf{v}) = 0$ corresponding to $\mathbf{x} = \mathbf{x}_A^\star$.

Compared to Pb_1, Pb_2 is the minimization of the same cost function f_A^+, under the same constraint $\mathbf{c} = 0$, but over a restricted admissible domain. However, the solution to Pb_1 belongs to this smaller admissible domain; therefore, it is also the solution to Pb_2.

Concerning the unconstrained problem Pb_3, since the cost function f_A^+ is convex, only the stationarity condition need be established. In this subproblem, $\mathbf{u} = 0$, and the sole unknown is $\mathbf{v}$. Letting $\mathbf{v} = (v_p, \dots, v_1)^t$, the partial derivatives to be examined are the following ones, to be evaluated for $\mathbf{v} = 0$:

$$\frac{\partial f_A^+(\mathbf{X}(0,\mathbf{v}))}{\partial v_j} = \nabla f_A^{+\star} . \frac{\partial \mathbf{X}(0,\mathbf{v})}{\partial v_j} = \nabla f_A^\star . [\mathbf{\Omega}]^{n-j+1} \tag{65}$$

for all j ($1 \leq j \leq p$), where $[\mathbf{\Omega}]^{n-j+1}$ denotes the $n-j+1st$ column vector of matrix $\mathbf{\Omega}$. On one hand, due to the optimality condition (46), $\nabla f_A^\star \in Sp\,(\nabla c_1^\star, \dots, \nabla c_K^\star) = \mathrm{Ker}\,\mathbf{P}$ which is a subspace of U; U itself is spanned by the first $n-p$ eigenvectors of matrix $\mathbf{H}_A^{+'}$. On the other hand, $n - j + 1 > n - p$. Hence the column vector $[\mathbf{\Omega}]^{n-j+1}$ is also an eigenvector of this matrix, but of superior index, therefore not in U and orthogonal to it. Thus , as a consequence of the orthogonality of matrix $\mathbf{\Omega}$, the above scalar product is equal to 0, and the gradient w.r.t. $\mathbf{v}$ is null:

$$\text{For } \mathbf{v} = 0: \quad \nabla_{\mathbf{v}} f_A^+ = 0. \tag{66}$$

□

Continuum of Nash equilibria. We now make the following:

Proposition 6
Under the hypotheses of Proposition 5, and the bounds specified in (56), if the convexity fix constant c is large enough, the Nash equilibrium $\bar{\mathbf{x}}_\varepsilon$ exists for all sufficiently small ε.

Proof: the following theorem is known [1] (p. 268):

Theorem 1 (Nash)
Suppose that the multistrategy set
(i) $X(N)$ is a convex compact subset
and that, for each player i, the loss function
(ii) f_i is continuous and $f_i(., \mathbf{x}^i)$ is convex for all $\mathbf{x}^i \in X^i$.
Then there exists a non-cooperative equilibrium.

Here, the strategy subsets of the two players are:

$$X^A = \left\{ \mathbf{u} \in \mathsf{R}^{n-p} \ / \ \|\mathbf{u}\| \le B_{\mathbf{u}} \right\}, \quad X^B = \{ \mathbf{v} \in \mathsf{R}^p \ / \ \|\mathbf{v}\| \le B_{\mathbf{v}} \}. \tag{67}$$

These are finite-dimensional closed balls, thus indeed convex compact subsets.

Player A, in effect, minimizes the Lagrangian L of (50) w.r.t. its strategy $\mathbf{u}$. By the adjustment of the convexity-fix constant c, this function is strictly convex, and so is also the corresponding partial function of this player's strategy.

Player B minimizes the cost function f_{AB} of (58) w.r.t. its strategy $\mathbf{v}$. This function is strictly-convex for all sufficiently small ε, and so is also the corresponding partial function of this player's strategy.

Hence Theorem 1 applies, and this establishes Proposition 6. □

We further make the

Hypothesis 2
The Nash equilibrium $\bar{\mathbf{x}}_\varepsilon$ depends smoothly on ε.

Then, $\{\bar{\mathbf{x}}_\varepsilon\}$ defines, as ε varies, a smooth continuum of Nash equilibria that originates from the single-discipline optimizing point $\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$.

Remark 1
At the origin of the continuum of Nash equilibria, $(\bar{\mathbf{u}}_0, \bar{\mathbf{v}}_0) = 0$ and the bounds in (56) are not active. Since continuity is assumed, this remains true for small enough ε. In this sense, these bounds are somewhat artificial.

Variations along the continuum of Nash equilibria. One can follow the variations of the various cost functions along the continuum of Nash equilibria by examining the following functions of ε:

$$\phi_A(\varepsilon) = f_A(\bar{\mathbf{x}}_\varepsilon), \ \phi_B(\varepsilon) = f_B(\bar{\mathbf{x}}_\varepsilon), \text{ and } \phi_{AB}(\varepsilon) = f_{AB}(\bar{\mathbf{x}}_\varepsilon) = \phi_A(\varepsilon) + \varepsilon\Big(\theta\phi_B(\varepsilon) - \phi_A(\varepsilon)\Big). \tag{68}$$

Along the continuum, the constraints are satisfied uniformly:

$$\forall k, \ \forall \varepsilon, \ c_k(\bar{\mathbf{x}}_\varepsilon) = 0. \tag{69}$$

Differentiating this w.r.t. ε, setting $\varepsilon = 0$, and recalling that $\bar{\mathbf{x}}_0 = \mathbf{x}_A^\star$ (by consistency) give:

$$\forall k, \ \nabla c_k^\star . \bar{\mathbf{x}}_0' = 0 \tag{70}$$

where $\bar{\mathbf{x}}_0' = \frac{d\bar{\mathbf{x}}_\varepsilon}{d\varepsilon}$ at $\varepsilon = 0$. This result is injected into the optimality condition (46) yielding

$$\phi_A'(0) = \nabla f_A^\star . \bar{\mathbf{x}}_0' = 0. \tag{71}$$

This proves that along the continuum, at least initially, the cost function f_A is maintained to quasi-optimality:

$$\phi_A(\varepsilon) = 1 + O(\varepsilon^2). \tag{72}$$

Lastly, from the definition of $\phi_{AB}(\varepsilon)$, $\phi'_{AB}(0) = 0 + 1.(\theta \times 1 - 1) + 0 \times (\dots)'$, that is:

$$\phi'_{AB}(0) = \theta - 1 \leq 0. \tag{73}$$

Equations (61), (72) and (73) summarize our first theoretical findings: in variable-space, at start of the continuation formulation ($\varepsilon = 0$), the Nash equilibrium solution exists and it is equal to the single-discipline (A) optimum solution; in function space, assuming under-relaxation ($\theta < 1$), the auxiliary cost function $\phi_{AB}(\varepsilon)$ diminishes initially, while $\phi_A(\varepsilon)$ necessarily increases. However, in practice, we usually set θ to 1, and obtain for well-behaved (antagonistic) problems the desirable effect of initially monotone-decreasing $\phi_{AB}(\varepsilon)$ and $\phi_B(\varepsilon)$, at the cost of a degradation of $\phi_A(\varepsilon)$ that is second-order in ε. This result is justified by enhancing the theory in the next section.

In conclusion, under a strict, but local convexity assumption that can be arranged by a simple fix, the proposed Nash game formulation offers a framework to reduce the auxiliary and secondary cost functions, f_{AB} and f_B, from their values at the point $\mathbf{x}_A^\star$ of optimality of the primary cost function f_A, while maintaining the quasi-optimality of the latter. By continuity, for sufficiently small ε, the equilibrium solution vector $\bar{\mathbf{x}}_\varepsilon$ is as close to $\mathbf{x}_A^\star$, and the cost function $\phi_A(\varepsilon)$ as close to $f_A^\star$ as desirable, and this provides confidence in both existence and physical relevance of the equilibrium solutions.

Remark 2
The territory splitting is defined through the diagonalization of the reduced Hessian matrix, $\mathbf{H}'_A$ (or $\mathbf{H}_A^{+'}$) which itself depends on the Hessian matrix $\mathbf{H}_A^\star$ of the agglomerated cost function $f_A(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_A^\star$, and the constraint gradients $\{\nabla c_k^\star\}$. These differential elements are often computed with a certain accuracy defined, say, by errors of order δ. We may question how is the territory splitting affected by δ. It is well-known from "perturbation theory" [22] (Chapter 2) that when the base matrix is real-symmetric, and the perturbations as well, the eigenvalue problem is well-conditioned. However, this may not be the case of the associated eigenvector problem, which here, plays a crucial role in the definition of the splitting. In particular, the expression of the first variation of the eigenvectors involves differences of eigenvalues in denominators. Hence one should be cautious when the spectrum is dense. A numerical illustration of this phenomenon is given in Appendix A. Consequently, we recommend, whenever possible, to select the dimension p which controls the players' strategies, in a manner to best separate the eigenvalues h'_{n-p} and h'_{n-p+1} of (54) associated with the last mode retained in Player A's strategy and the first mode in Player B's strategy respectively.

4.2 Re-examination of the convexity fix on the primary function

In some cases, the primary optimization problem, that is, the minimization of function f_A under the constraint $\mathbf{c} = 0$, is well-posed while f_A is not locally convex. In such a situation, we have proposed to substitute the augmented cost function $f_A^+(\mathbf{x})$ of (48) to $f_A(\mathbf{x})$. In this case, the territory splitting is constructed according to the eigenstructure of the augmented matrix $\mathbf{H}_A^{+'}$ of (52), and we may question how the splitting is affected by this fix.

Suppose that ξ is an eigenvector of matrix $\mathbf{H}'_A$ associated with the possibly-negative eigenvalue λ:

$$\mathbf{H}'_A \xi = \mathbf{P}\mathbf{H}_A^\star \mathbf{P}\xi = \lambda \xi. \tag{74}$$

Then: either $\xi \in \operatorname{Ker} \mathbf{P}$ and ξ is also and eigenvector of $\mathbf{H}_A^{+'}$ associated with the eigenvalue $\lambda = 0$, or $\xi \in Im(\mathbf{P}) = \operatorname{Ker} \mathbf{P}^\perp$; in the latter case, $\mathbf{P}\xi = \xi$ and

$$\mathbf{H}_A^{+'} \xi = (\lambda + c)\xi. \tag{75}$$

Hence, the proposed convexity-fix has no effect on the eigenvector structure associated with the augmented cost function, as well as on their conventional ordering, since the eigenvalues associated with those in $Im(\mathbf{P})$ are all augmented of the same constant c. Thus, the territory splitting *per se* is not affected by the convexity fix. However, resorting to a convex primary cost function by the proposed substitution is necessary in the definition of the the Nash game, to insure theoretically the existence of the equilibrium as well as the convergence of the numerical process.

4.3 Examples

This first set of examples involve a single primary discipline ($m = 1$), one scalar constraint ($K = 1$), and one secondary discipline ($M = 2$ in total). To better follow the variations of functions woth ε along the continuum, we let:

$$\phi_A(\varepsilon) = f_A^+(\overline{\mathbf{x}}_\varepsilon),\ \phi_B(\varepsilon) = f_B(\overline{\mathbf{x}}_\varepsilon),\ \phi_{AB}(\varepsilon) = f_{AB}(\overline{\mathbf{x}}_\varepsilon) = \phi_A(\varepsilon) + \varepsilon\big(\theta\,\phi_B(\varepsilon) - \phi_A(\varepsilon)\big). \tag{76}$$

<u>**Test-case 1:**</u> **Simple case involving convex quadratic functions.** Are considered: $n = 4$ variables, split into $p = 2$ variables assigned to *Player B*'s strategy, and $n - p = 2$ variables assigned to *Player A*'s strategy; functions:

$$f_A = f_1 = \sum_{i=1}^{4} \frac{x_i^2}{3^i};\ \mathbf{c} = c_1 = x_1^4 x_2^3 x_3^2 x_4 - 96\sqrt{3};\ f_B = f_2 = \sum_{i=1}^{4} x_i^2. \tag{77}$$

The variations of the functions $\phi_A(\varepsilon)$, $\phi_{AB}(\varepsilon)$ and $\phi_B(\varepsilon)$ is represented on Figure 5a. Due to the nonlinear constraint, although the auxiliary function $\phi_{AB}(\varepsilon)$ is monotone-decreasing throughout, it is not the case of $\phi_B(\varepsilon)$ which reaches a minimum at $\varepsilon \approx 0.487$; no advantage could be drawn from using larger values of ε.

<u>**Test-case 2:**</u> **Simple case in which the primary cost function is concave.** Are considered: $n = 3$ variables, split into $p = 1$ variable assigned to *Player B*'s strategy, and $n - p = 2$ variables assigned to *Player A*'s strategy; functions:

$$f_A = f_1 = 3 - (x_1^2 + x_2^2 + x_3^2 + x_1);\ \mathbf{c} = c_1 = x_1^2 + x_2^2 + x_3^2 - 1;\ f_B = f_2 = (1 - x_3)^2. \tag{78}$$

Here the primary cost function is uniformly concave. However the primary optimization problem (minimization of f_A under the constraint $\mathbf{c} = 0$) is well-posed and admits the unique solution $\mathbf{x}_A^\star = (1, 0, 0)$. Thus, in the Nash game, the cost function f_A is replaced by the following:

$$f_A^+ = f_A + \frac{c}{2}\left[(x_1 - 1)^2 + x_2^2 + x_3^2\right] \tag{79}$$

that admits the same minimum at the same point $\mathbf{x}_A^\star$, and is uniformly convex provided that $c > 2$. Let us set $c = 4$. Then the test-case can be fully worked out formally.

Player A's strategy must contain the variable x_1, and include a variable associated with an orthogonal axis, actually arbitrary. Let us choose x_2. Then, for Player B's strategy, only remains x_3 corresponding to the sole axis orthogonal to *Player A*'s strategy. One has:

$$f_A^+ = 5 + \sum_{i=1}^{3} x_i^2 - 5x_1,\ f_{AB} = (1 - \varepsilon) f_A^+ + \varepsilon f_B. \tag{80}$$

The unconstrained minimization of f_{AB} w.r.t. x_3 for fixed (x_1, x_2) is realized by the condition $\partial f_{AB}/\partial x_3 = 0$, and this writes $2x_3 - 2\varepsilon = 0$. Hence:

$$x_3 = \bar{x}_{3,\varepsilon} = \varepsilon. \tag{81}$$

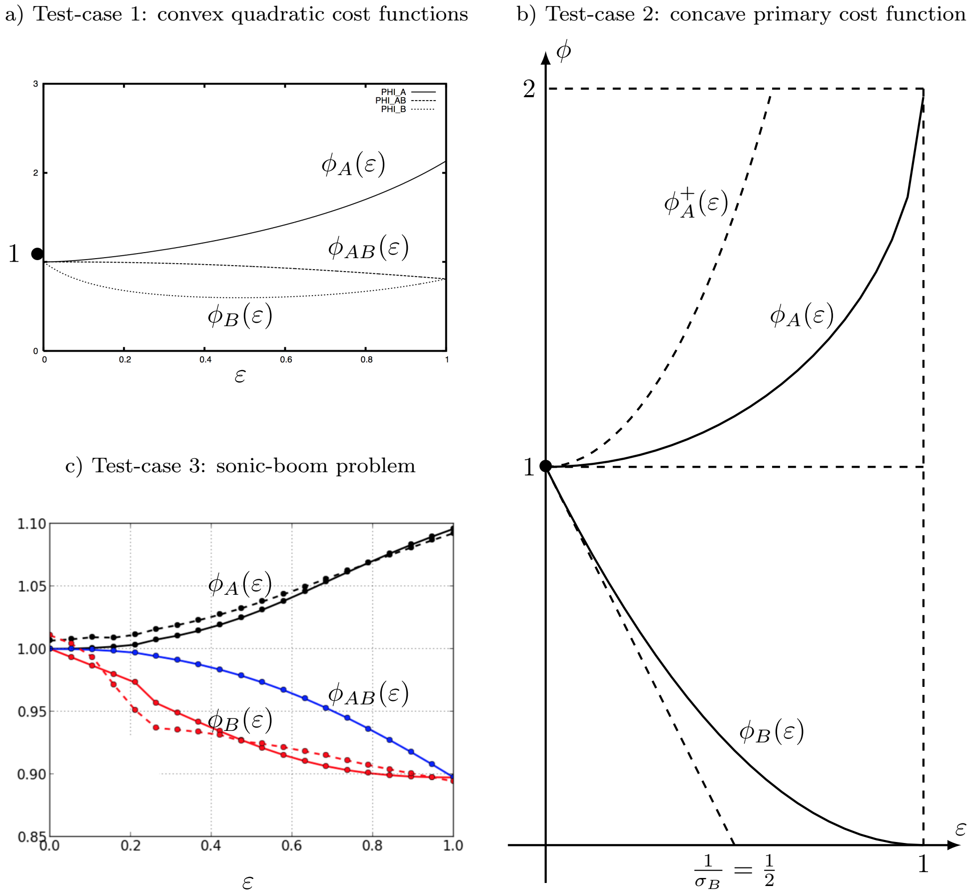

Figure 5: Test-cases 1, 2 and 3: cost functions along the continuum of Nash equilibria. (In test-case 3, solid lines are associated with meta-models, and dotted lines with *a posteriori* finite-volume three-dimensional flow computations.)

Then, the constrained minimization of f_A^+ w.r.t. (x_1, x_2) for fixed x_3 consists in assigning to x_1 the largest value left possible for the satisfaction of the constraint after the assignment of x_3, and that is:

$$x_1 = \bar{x}_{1,\varepsilon} = \sqrt{1-\varepsilon^2}, \quad x_2 = \bar{x}_{2,\varepsilon} = 0. \tag{82}$$

Evidently, the continuum of Nash equilibria is the circular arc connecting the starting point $\mathbf{x}_A^\star$ with the pole (0,0,1) and defined analytically by: $x_1^2 + x_3^2 = 1$ $(0 \leq x_3 \leq 1)$ and $x_2 = 0$. It should be noted that this arc does not depend on the value of the constant c introduced to consider a convex primary function. In summary, along the arc, one has:

$$\begin{cases} \bar{\mathbf{x}}_\varepsilon = (\sqrt{1-\varepsilon^2}, 0, \varepsilon) \\ \phi_A(\varepsilon) = f_A(\bar{\mathbf{x}}_\varepsilon) = 2 - \sqrt{1-\varepsilon^2} = 1 + \dfrac{\varepsilon^2}{2} + O(\varepsilon^4) \\ \phi_A^+(\varepsilon) = f_A^+(\bar{\mathbf{x}}_\varepsilon) = 6 - 5\sqrt{1-\varepsilon^2} = 1 + \dfrac{5\varepsilon^2}{2} + O(\varepsilon^4) \\ \phi_B(\varepsilon) = f_B(\bar{\mathbf{x}}_\varepsilon) = (1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \end{cases} \tag{83}$$

(see Figure 5b).

This test-case illustrates the fact that the absence of convexity of the primary cost function is not a barrier to the application of our splitting technique.

<u>**Test-case 3:**</u> **Drag minimization and sonic-boom reduction in 3D aerodynamics.** Several examples of application of Nash games with the present territory splitting have been provided by the theses [15], [18] and [14], that involve optimum-shape design in compressible aerodynamics as primary discipline, concurrently with another discipline, treated as secondary. These examples have been reported in some details in [10]. The aero-structural wing-shape optimization [15] was the first of a series and raised the most acute antagonism between two disciplines and motivated our original theoretical and numerical development of Nash games with territory splitting. Here, for sake of brevity, we restrict ourselves to a numerical experiment from [18], courtesy of A. Minelli, related to the shape optimization of a supersonic generic configuration w.r.t. drag in cruise conditions, subject to a lift constraint, concurrently with sonic-boom reduction.

The primary cost function $f_A = C_D$ (aerodynamic drag coefficient) and the scalar constraint function $\mathbf{c} = c_1 = C_L - C_{L_0}$ (C_L: aerodynamic lift coefficient; C_{L_0}: specification) were calculated by the three-dimensional Eulerian-flow simulation using the ONERA finite-volume code elsA [3]. The secondary cost function $f_B = \sum_k |\Delta p_k|$ was a measure of the sonic-boom intensity at ground level, as the sum of the N-shaped pressure jumps. This criterion was evaluated from near-field pressure by integration via a ray-tracing procedure. The trade-off between the two cost functions was again evaluated by the Nash game formulation.

The continuum of Nash equilibria was initiated from the optimum solution of the sole primary cost function. This continuum is represented on Figure 5c. Even though the framework of this numerical experiment is far more complex than optimization of analytical functions, all three convergence plots of Figure 5 demonstrate similar characteristics. This confirms the generality of the approach. In particular, most importantly: the continuum of Nash equilibria initiates at the primary-discipline optimum design-point as a consequence of the consistency result; additionally, as ε increases, the primary discipline is initially maintained to quasi-optimality, while the secondary cost function actually diminishes faster (linearly in ε). In the present numerical experiment, the continuum of Nash equilibria has been calculated via meta-models (solid lines), but the points have been evaluated *a posteriori* by the high-fidelity model of the 3D Euler equations. The small discrepancy at $\varepsilon = 0$ is due to a minor inconsistency in the meta-model; this has no serious consequence.

Lastly, we point out that this experiment was realized without convexity-fix ($c = 0$).

5 Nash game for the prioritized multi-objective problem

At present, we generalize the theory of territory splitting of Section 4 to problems involving more than two disciplines, namely a whole set of prioritized cost functions, $\{f_j(\mathbf{x})\}$ $(j = 1, \dots, m)$, where possibly $m > 1$, but also a whole set of secondary cost functions $\{f_j(\mathbf{x})\}$ $(j = m+1, \dots, M)$ of lesser importance, where possibly $M > m+1$. Again, known properties of the Multiple-Gradient Descent Algorithm (MGDA) are used to globalize the secondary cost functions in a single function, $f_B(\mathbf{x})$, and the theory of Section 4 is applied, and in fact enhanced. Thus here the cost functions $f_A(\mathbf{x})$ and $f_B(\mathbf{x})$ are not part of the problem formulation, but constructed from more general lists.

5.1 Theory

Again it is recalled that if the cost function f_A of (47) or the Lagrangian L of (50) is not strictly-convex at $\mathbf{x} = \mathbf{x}_A^\star$, they should be augmented by the convexity-fix term $\frac{c}{2}\left\|\mathbf{x} - \mathbf{x}_A^\star\right\|^2$, prior to all subsequent developments.

One considers the continuum of Nash equilibria between the primary cost function f_A^+ and the cost function f_{AB} of (58), except that now, the cost function f_B is not specified by the problem formulation, but is to be defined precisely according to a specific algorithmic choice that is made below. This definition is meant to be such that the cost function f_B correctly represents the entire family of secondary cost functions $\{f_j\}$ $(j = m+1, \dots, M)$ along the continuum, at least for small ε. For the moment, f_B is defined as the following convex combination of the secondary cost functions

$$f_B = \sum_{j=m+1}^{M} \alpha_j^\star \frac{f_j}{f_j^\star}, \tag{84}$$

where $f_j^\star = f_j(\mathbf{x}_A^\star)$ so that $f_B^\star = f_B(\mathbf{x}_A^\star) = 1$, and the new coefficients $\{\alpha_j^\star\}$ $(j > m)$ are left free for the moment but such that $\alpha_j^\star \geq 0$ $(\forall j)$ and $\sum_{j=m+1}^{M} \alpha_j^\star = 1$.

The variations of several functions along the continuum are examined. Thus (55) holds, and the following notations are also used:

$$\boldsymbol{\Omega} = (\boldsymbol{\Omega}_{\mathbf{u}}\ \boldsymbol{\Omega}_{\mathbf{v}}), \quad \boldsymbol{\Omega}_{\mathbf{u}} = \begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{uu}} \\ \boldsymbol{\Omega}_{\mathbf{vu}} \end{pmatrix}, \quad \boldsymbol{\Omega}_{\mathbf{v}} = \begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{uv}} \\ \boldsymbol{\Omega}_{\mathbf{vv}} \end{pmatrix}, \tag{85}$$

where the diagonal blocks $\boldsymbol{\Omega}_{\mathbf{uu}}$ and $\boldsymbol{\Omega}_{\mathbf{vv}}$ are $(n-p) \times (n-p)$ and $p \times p$ respectively, so that

$$\mathbf{x} = \mathbf{x}_A^\star + \boldsymbol{\Omega}_{\mathbf{u}}\mathbf{u} + \boldsymbol{\Omega}_{\mathbf{v}}\mathbf{v} \tag{86}$$

is equivalent to (55). Throughout, the symbols $\nabla_{\mathbf{u}}$ and $\nabla_{\mathbf{v}}$ are used for partial gradients w.r.t. $\mathbf{u}$ (for fixed $\mathbf{v}$) and $\mathbf{v}$ (for fixed $\mathbf{u}$); these are related to the full gradients by:

$$\nabla_{\mathbf{u}}(.) = \boldsymbol{\Omega}_{\mathbf{u}}^t \nabla(.), \quad \nabla_{\mathbf{v}}(.) = \boldsymbol{\Omega}_{\mathbf{v}}^t \nabla(.). \tag{87}$$

These operators produce vectors of dimension $n-p$ and p respectively. Additionally, define the partial Hessian of f_A^+ w.r.t. $\mathbf{v}$ (for fixed $\mathbf{u}$), evaluated at $\mathbf{x}_A^\star$:

$$\mathbf{S} = \nabla_{\mathbf{vv}}^2 f_A^{+\star}. \tag{88}$$

This $p \times p$ symmetric matrix is positive-definite by convexity of f_A^+. In fact, it is a diagonal block. To see this, note that

$$\mathbf{S} = \boldsymbol{\Omega}_{\mathbf{v}}^t \mathbf{H}_A^{+\star} \boldsymbol{\Omega}_{\mathbf{v}} = \boldsymbol{\Omega}_{\mathbf{v}}^t \Big[\mathbf{P} + (\mathbf{I}_n - \mathbf{P})\Big] \mathbf{H}_A^{+\star} \Big[\mathbf{P} + (\mathbf{I}_n - \mathbf{P})\Big] \boldsymbol{\Omega}_{\mathbf{v}}. \tag{89}$$

Then observe that:

$$(\mathbf{I}_n - \mathbf{P})\boldsymbol{\Omega}_{\mathbf{v}} = 0_{n\times(n-p)}, \quad \boldsymbol{\Omega}_{\mathbf{v}}^t(\mathbf{I}_n - \mathbf{P}) = 0_{(n-p)\times n}, \tag{90}$$

since $(\mathbf{I}_n - \mathbf{P})$ is the matrix associated with the orthogonal projection onto the subspace spanned by the contraint gradients and the column-vectors of matrix $\boldsymbol{\Omega}_{\mathbf{v}}$ are orthogonal to them. Hence, (89) simplifies to:

$$\mathbf{S} = \boldsymbol{\Omega}_{\mathbf{v}}^t \mathbf{P} \mathbf{H}_A^{+\star} \mathbf{P} \boldsymbol{\Omega}_{\mathbf{v}} = \boldsymbol{\Omega}_{\mathbf{v}}^t \mathbf{H}_A^{+\prime} \boldsymbol{\Omega}_{\mathbf{v}} = \boldsymbol{\Omega}_{\mathbf{v}}^t \boldsymbol{\Omega} \mathcal{H} \boldsymbol{\Omega}^t \boldsymbol{\Omega}_{\mathbf{v}} = \mathcal{H}_{\mathbf{v}} \tag{91}$$

in which $\mathcal{H}_{\mathbf{v}}$ is the "$\mathbf{v}$ diagonal block", that is, the lower $p \times p$ diagonal block of the positive-definite diagonal matrix $\mathcal{H}$.

We also define the new variable

$$\mathbf{w} = \mathbf{S}^{\frac{1}{2}}\mathbf{v}. \tag{92}$$

and use the symbol $\nabla_{\mathbf{w}}$ for the partial gradient w.r.t. $\mathbf{w}$ (for fixed $\mathbf{u}$) so that

$$\nabla_{\mathbf{w}}(.) = \mathbf{S}^{-\frac{1}{2}} \nabla_{\mathbf{v}}(.) = \mathbf{S}^{-\frac{1}{2}} \boldsymbol{\Omega}_{\mathbf{v}}^t \nabla(.). \tag{93}$$

On one hand, one begins by examining the orders of magnitude of $\mathbf{u}$ and $\mathbf{v}$ that result from the Nash game in the limit $\varepsilon \to 0$. It is assumed that by regularity, one has at least

$$\mathbf{u} = O(\varepsilon), \quad \mathbf{v} = O(\varepsilon). \tag{94}$$

Figure 6 provides a sketch of the geometrical configuration of the continuum of Nash equilibria $\{\bar{\mathbf{x}}_\varepsilon\}$, lying on the constraint manifold $\mathbf{c} = 0$, in the vicinity of the starting point $\mathbf{x}_A^\star = \bar{\mathbf{x}}_0$. By construction of the split:

– Both points $\mathbf{x}_A^\star$ and $\bar{\mathbf{x}}_\varepsilon$ belong to the constraint manifold, and $\bar{\mathbf{x}}_\varepsilon \to \mathbf{x}_A^\star$ as $\varepsilon \to 0$.

– Matrix $\mathbf{\Omega}$ is orthogonal and the vectors $\mathbf{\Omega_u u}$ and $\mathbf{\Omega_v v}$ are the generic vectors of the supplementary subspaces $Sp\left(\mathbf{q}^1, \dots, \mathbf{q}^{n-p}\right)$ and $Sp\left(\mathbf{q}^{n-p+1}, \dots, \mathbf{q}^n\right)$ respectively.

– Since $n - p > K$, the subspace $Sp\left(\mathbf{q}^1, \dots, \mathbf{q}^{n-p}\right)$ contains $Sp\left(\mathbf{q}^1, \dots, \mathbf{q}^K\right) = Sp\left(\nabla c_1^\star, \dots, \nabla c_K^\star\right)$; but the constraint gradients are the normals to the manifold; hence, $\mathbf{\Omega_v v}$ is tangent to the manifold, and $\mathbf{\Omega_u u}$ orthogonal to it at $\mathbf{x} = \mathbf{x}_A^\star$.

The constraint manifold is assumed to be locally smooth. Let $\mathbf{x}_{AE}$ be the vector connecting the point $\mathbf{x}_A^\star$ to $\bar{\mathbf{x}}_\varepsilon$. The three vectors $\mathbf{x}_{AE}$, $\mathbf{\Omega_v v}$ and $\mathbf{\Omega_u u}$ are coplanar and form a rectangle triangle. Let τ be the angle between vector $\mathbf{x}_{AE}$ (the hypothenuse, "a chord" for the manifold) and vector $\mathbf{\Omega_v v}$ (tangent to the manifold). The angle τ is infinitesimally small in the limit $\varepsilon \to 0$, and for a smooth surface $\tau = O(\varepsilon)$. Then, since

$$\|\mathbf{\Omega_u u}\| = \|\mathbf{\Omega_v v}\| \tan \tau \tag{95}$$

(94) can readily be enhanced as follows:

$$\mathbf{u} = O(\varepsilon^2), \quad \mathbf{v} = O(\varepsilon). \tag{96}$$

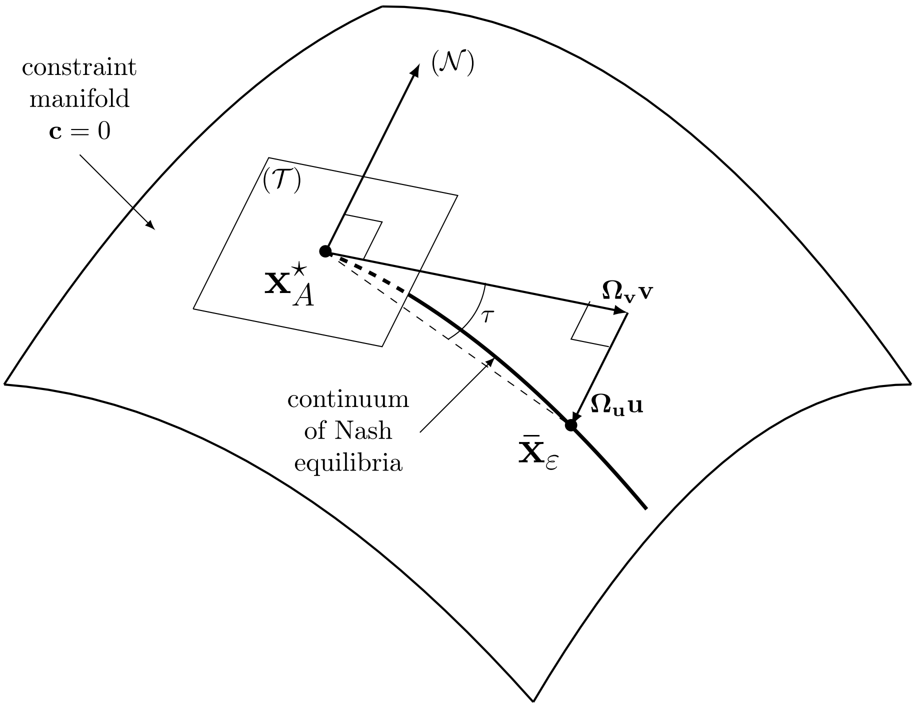

Figure 6: Geometrical configuration of the continuum of Nash equilibria $\{\bar{\mathbf{x}}_\varepsilon\}$ in the vicinity of the starting point $\mathbf{x}_A^\star$

Remark 3
This result describes how the Nash game does preserve the optimum of the primary cost function to second-order in ε: vector $\mathbf{u}$ alters f_A^+ and f_A by a term $O(\varepsilon^2)$, whereas vector $\mathbf{v}$ has no effect on it (to second-order) since $\mathbf{\Omega_v v} \perp \nabla {f_A^+}^\star = \nabla f_A^\star$.

On the other hand, $\mathbf{v}$ is the solution of Problem Pb_3, that is, it minimizes the cost function f_{AB} of (58) w.r.t. $\mathbf{v}$ (for fixed $\mathbf{u}$). But:

$$f_{AB}(\mathbf{v}) = f_{AB}(0) + \nabla_\mathbf{v} f_{AB}.\mathbf{v} + \tfrac{1}{2}\mathbf{v}.\nabla^2_{\mathbf{vv}} f_{AB}\mathbf{v} + O(\varepsilon^3) \tag{97}$$

so that

$$\nabla_{\mathbf{v}} f_{AB} + \nabla^2_{\mathbf{vv}} f_{AB}\mathbf{v} + O(\varepsilon^2) = 0 \tag{98}$$

which gives

$$\mathbf{v} = -(\nabla^2_{\mathbf{vv}} f_{AB})^{-1}\nabla_{\mathbf{v}} f_{AB} + O(\varepsilon^2). \tag{99}$$

Then:

$$\nabla_{\mathbf{v}} f_{AB} = (1-\varepsilon)\nabla_{\mathbf{v}} f_A^+ + \varepsilon\theta\,\nabla_{\mathbf{v}} f_B. \tag{100}$$

But:

$$f_A^+ = 1 + \varepsilon^2 h_A \tag{101}$$

where h_A remains finite in the limit $\varepsilon \to 0$, and the first term in (100) is $O(\varepsilon^2)$. Hence, as $\varepsilon \to 0$:

$$\nabla_{\mathbf{v}} f_{AB} = -\varepsilon\theta\nabla_{\mathbf{v}} f_B^\star + O(\varepsilon^2). \tag{102}$$

Additionally:

$$\nabla^2_{\mathbf{vv}} f_{AB} = (1-\varepsilon)\nabla^2_{\mathbf{vv}} f_A^+ + \varepsilon\theta\,\nabla^2_{\mathbf{vv}} f_B \to \left(\nabla^2_{\mathbf{vv}} f_A^+\right)^\star = \mathbf{S}. \tag{103}$$

These results are injected in (99) to get the following estimate for $\mathbf{v}$, linear in ε:

$$\mathbf{v} = -\varepsilon\theta\mathbf{S}^{-1}\nabla_{\mathbf{v}} f_B^\star + O(\varepsilon^2). \tag{104}$$

We now examine the variations along the continuum of the secondary cost functions by letting:

$$\phi_j(\varepsilon) = \frac{f_j(\bar{\mathbf{x}}_\epsilon)}{f_j^\star} \quad (j = m+1,\dots,M). \tag{105}$$

One has:

$$\phi_j(\varepsilon) - 1 = \frac{1}{f_j^\star}\nabla f_j^\star . \big(\boldsymbol{\Omega}_{\mathbf{u}}\mathbf{u} + \boldsymbol{\Omega}_{\mathbf{v}}\mathbf{v}\big) + O(\varepsilon^2) = O(\varepsilon^2) - \frac{\varepsilon\theta}{f_j^\star}\nabla f_j^\star . \boldsymbol{\Omega}_{\mathbf{v}}\mathbf{S}^{-1}\nabla_{\mathbf{v}} f_B^\star \tag{106}$$

and this provides the derivative:

$$\phi_j'(0) = -\frac{\theta}{f_j^\star}\nabla f_j^\star . \boldsymbol{\Omega}_{\mathbf{v}}\mathbf{S}^{-1}\nabla_{\mathbf{v}} f_B^\star = -\frac{\theta}{f_j^\star}(\nabla_{\mathbf{v}} f_B^\star)^t\mathbf{S}^{-1}\nabla_{\mathbf{v}} f_j^\star = -\theta(\nabla_{\mathbf{w}} f_B^\star)^t\nabla_{\mathbf{w}}(f_j/f_j^\star). \tag{107}$$

Lastly, we introduce again the definition and properties of the MGDA construction. Consider the following logarithmic gradients:

$$\mathbf{g}_j = \nabla_{\mathbf{w}}\left(f_j/f_j^\star\right) = \mathbf{S}^{-\frac{1}{2}}\boldsymbol{\Omega}_{\mathbf{v}}^t\nabla\left(f_j/f_j^\star\right) \qquad (j = m+1,\dots,M). \tag{108}$$

here taken w.r.t. the newly-defined vector $\mathbf{w}$. Note that for the minimization of the secondary cost functions, the above relevant gradient vectors do not fall in the definition adopted for the primary optimization problem, (44).

Let $\boldsymbol{\omega}_B^\star$ be the element of minimum Euclidean norm in the convex hull of these gradients. Such a vector is unique. Assume $\boldsymbol{\omega}_B^\star \neq 0$, that is, assume that the vector $\{\mathbf{g}_j\}$ are not in a configuration of Pareto-stationarity. Then, the vector $\boldsymbol{\omega}_B^\star$ can be expressed as the following convex combination:

$$\boldsymbol{\omega}_B^\star = \sum_{j=m+1}^{M}\alpha_j^\star\mathbf{g}_j \tag{109}$$

where $\alpha_j^\star \geq 0$ $(\forall j)$, and $\sum_{j=m+1}^{M}\alpha_j^\star = 1$. The coefficients $\{\alpha_j^\star\}$ are not necessarily unique in case the vectors $\{\mathbf{g}_j\}$ are linearly dependent. This setting completes the definition of f_B in (84). It follows that:

$$\boldsymbol{\omega}_B^\star = \nabla_{\mathbf{w}} f_B. \tag{110}$$

Consider the following subsets of indices associated with the representation of $\boldsymbol{\omega}_B^\star$, (109):

$$J_0 = \{\, j \in \mathbb{N} \;/\; m+1 \leq j \leq M;\ \alpha_j = 0\,\}, \quad J^\star = \{\, j \in \mathbb{N} \;/\; m+1 \leq j \leq M;\ \alpha_j \neq 0\,\}. \tag{111}$$

The MGDA construction is such that the derivatives of the cost functions $\{f_j/f_j^\star\}$ $(j \in J^\star)$ in the direction of $\boldsymbol{\omega}_B^\star$ are equal to some strictly-positive constant σ_B:

$$\mathbf{g}_j.\boldsymbol{\omega}_B^\star = \nabla_{\mathbf{w}} \left(f_j/f_j^\star\right).\nabla_{\mathbf{w}} f_B = \|\boldsymbol{\omega}_B^\star\|^2 = \sigma_B \quad (\forall j \in J^\star) \tag{112}$$

and this gives:

$$\phi_j'(0) = -\theta\sigma_B \quad (\forall j \in J) \tag{113}$$

and by convex combination, this is also the derivative $\phi_B'(0)$ of $\phi_B(\varepsilon) = f_B(\bar{\mathbf{x}}_\varepsilon)$ at $\varepsilon = 0$.

For indices $j \in J_0$ if any:

$$\mathbf{g}_j.\boldsymbol{\omega}_B^\star = \nabla_{\mathbf{w}} \left(f_j/f_j^\star\right).\nabla_{\mathbf{w}} f_B > \sigma_B \quad (\forall j \in J_0) \tag{114}$$

and this gives:

$$\phi_j'(0) < -\theta\sigma_B \quad (\forall j \in J_0). \tag{115}$$

Equations (113) and (115) conclude our analysis: the original objective to steer the entire family of secondary cost functions by the sole cost function f_B, at least for small ε, has indeed been achieved. Additionally, the condition under which the Nash game is proved to be effective in reducing the secondary cost functions has been identified: the gradients of the secondary cost functions w.r.t. to the variable $\mathbf{w}$ defined in (108) must not be in a configuration of Pareto-stationarity. This condition does not require the application of under-relaxation ($\theta < 1$). Therefore, in all our subsequent numerical experiments θ is set to 1.

Remark 4
If instead, the reduced vectors $\{\mathbf{g}_j\}$ are found in a configuration of Pareto-stationarity, the problem is ill-posed. If $\|\boldsymbol{\omega}_B^\star\|$ or σ_B is small but nonzero, it is recommended, if possible, to increase the dimension p of subspace V and reformulate the Nash game.

Remark 5
Of course, (113) applies in particular to the two-discipline case ($m = 1$, $M = 2$) for which $f_A = f_1/f_1^\star$ and $f_B = f_2/f_2^\star$, as it can be verified in the numerical experiments reported in Figure 5. More importantly, note that in this case, (113) enhances the result of (73), since bounds on the derivatives w.r.t. ε are established a priori.

Remark 6
The definition of the variable $\mathbf{w}$, (92), is informative of how the negotiation between the two virtual players does operate. Intuitively, one would guess that the secondary cost functions react according to their partial gradients w.r.t. $\mathbf{v}$ since this vector constitutes Player B's strategy. However, this is not exactly so. In effect, these partial gradients are scaled by the inverse square root of the block $\mathbf{S}$, and this reflects the influence of Player A's strategy on Player B.

5.2 Examples

<u>Test-case 4:</u> an example with $n = 4$ variables, a single primary cost function $f_A = f_1$ ($m = 1$), one scalar constraint ($K = 1$), and two secondary cost functions f_2 and f_3 ($M = 3$ disciplines in total).
We consider again the case of a concave primary cost function f_A to emphasize the necessity to

substitute to it an augmented cost function that is convex. The problem is defined as follows:

$$\begin{cases} \mathbf{x} = (x_1, x_2, x_3, x_4)^t \\ f_A(\mathbf{x}) = f_1 = 3 - \left(\sum_{i=1}^{4} x_i^2 + x_1 \right) \\ \mathbf{c}(\mathbf{x}) = c_1 = \sum_{i=1}^{4} x_i^2 - 1 \\ f_2(\mathbf{x}) = (x_3 - 1)^2 + \dfrac{4(x_4 - 1)^2}{r} - \dfrac{4}{r} + \dfrac{1 - x_1}{5} \\ f_3(\mathbf{x}) = -r(x_3 - 1)^2 + (x_4 - 1)^2 + r + 1 - x_1 \end{cases} \tag{116}$$

The free parameter r ($r > 0$) will be chosen later. Consequently: $\mathbf{x}_A^\star = (1, 0, 0, 0)^t$ and $f_A^\star = f_1^\star = f_2^\star = f_3^\star = 1$.

Since $f_A(\mathbf{x})$ is not convex at $\mathbf{x}_A^\star$ (in fact, uniformly concave), it is augmented by the convexity-fix term to give:

$$f_A^+(\mathbf{x}) = f_A(\mathbf{x}) + \frac{c}{2}\big(\mathbf{x} - \mathbf{x}_A^\star, \mathbf{x} - \mathbf{x}_A^\star\big) \tag{117}$$

where here the constant c must be greater than 2. Setting for example $c = 4$, one gets:

$$f_A^+(\mathbf{x}) = 5 + \sum_{i=1}^{4} x_i^2 - 5x_1 \tag{118}$$

Suppose we consider a split that assigns 2 variables to each player ($n - p = p = 2$). Here, one has:

$$\mathbf{H}_A^{+\star} = \nabla^2 f_A^{+\star} = 2\mathbf{I}_4 \tag{119}$$

and since $\nabla \mathbf{c}^\star = (2, 0, 0, 0)^t$:

$$\mathbf{P} = \mathbf{I}_4 - \frac{[\nabla \mathbf{c}^\star][\nabla \mathbf{c}^\star]^t}{[\nabla \mathbf{c}^\star]^t[\nabla \mathbf{c}^\star]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{120}$$

and:

$$\mathbf{H}_A^{+\prime} = \mathbf{P}\mathbf{H}_A^{+\star}\mathbf{P} = 2\mathbf{P}^2 = 2\mathbf{P}, \quad \mathcal{H}_\mathbf{u} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathcal{H}_\mathbf{v} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbf{I}_2. \tag{121}$$

Consequently, *Player A*'s strategy must be made of x_1 complemented by any other variable whose axis is orthogonal; one chooses x_2. Then, *Player B*'s strategy can be made of any two variables spanning the (x_3, x_4) plane and whose axes are orthogonal. One chooss (x_3, x_4). Hence:

$$\mathbf{\Omega} = \mathbf{I}_4, \quad \mathbf{u} = \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}. \tag{122}$$

Here $\mathbf{S} = 2\mathbf{I}_2$ and $\mathbf{w} = \sqrt{2}\mathbf{v}$. Thus, to define f_B, one examines the following reduced gradients:

$$\nabla_\mathbf{w} f_2^\star = \frac{1}{\sqrt{2}} \nabla_\mathbf{v} f_2^\star = \begin{pmatrix} \sqrt{2}(x_3^\star - 1) \\ \frac{4\sqrt{2}}{r}(x_4^\star - 1) \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -\frac{4\sqrt{2}}{r} \end{pmatrix} \tag{123}$$

$$\nabla_\mathbf{w} f_3^\star = \frac{1}{\sqrt{2}} \nabla_\mathbf{v} f_3^\star = \begin{pmatrix} -\sqrt{2}r(x_3^\star - 1) \\ \sqrt{2}(x_4^\star - 1) \end{pmatrix} = \begin{pmatrix} \sqrt{2}r \\ -\sqrt{2} \end{pmatrix}. \tag{124}$$

Evidently, for large r, the angle between these two vectors is indeed less than π but close to it, and this results in the unfavorable near Pareto-stationarity situation. In the limit $r \to \infty$, f_2 and f_3 could not be reduced simultaneously. Let us set $r = 4$ so that:

$$\nabla_\mathbf{w} f_2^\star = \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \quad \nabla_\mathbf{w} f_3^\star = \begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \end{pmatrix} \tag{125}$$

a) Proper split implementation

Clearly the minimum-norm element in the convex hull of the reduced gradients is:

$$\boldsymbol{\omega}_B^\star = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} = \frac{4}{5}\nabla_{\mathbf{w}} f_2^\star + \frac{1}{5}\nabla_{\mathbf{w}} f_3^\star. \tag{126}$$

It is associated with the common directional derivative

$$\sigma_B = \boldsymbol{\omega}_B^\star.\nabla_{\mathbf{w}} f_2^\star = \boldsymbol{\omega}_B^\star.\nabla_{\mathbf{w}} f_3^\star = 2. \tag{127}$$

Hence one is led to define:

$$f_B(\mathbf{x}) = \frac{4}{5}f_2(\mathbf{x}) + \frac{1}{5}f_3(\mathbf{x}) = (x_4-1)^2 + \frac{9}{25}(1-x_1) \tag{128}$$

and the auxiliary cost function is formed:

$$f_{AB}(\mathbf{x}) = (1-\varepsilon)f_A^+(\mathbf{x}) + \varepsilon f_B(\mathbf{x}) = (1-\varepsilon)\left(5 + \sum_{i=1}^4 x_i^2 - 5x_1\right) + \varepsilon(x_4-1)^2 + \frac{9\varepsilon}{25}(1-x_1). \tag{129}$$

On one hand, according to *Player B*'s strategy, the function $f_{AB}(\mathbf{x})$ must be minimized w.r.t. (x_3, x_4) for fixed (x_1, x_2) subject to no constraints. The result is immediate:

$$x_3 = \bar{x}_{3,\varepsilon} = 0, \quad x_4 = \bar{x}_{4,\varepsilon} = \varepsilon. \tag{130}$$

On the other hand, according to *Player A*'s strategy, (x_1, x_2) should be optimized to minimize $f_A^+(\mathbf{x})$ subject to the constraint and for fixed (x_3, x_4). This is realized by setting x_2 to 0 and x_1 to the largest value left possible for the satisfaction of the constraint after the assignment of (x_3, x_4), and that is:

$$x_1 = \bar{x}_{1,\varepsilon} = \sqrt{1-\varepsilon^2}, \quad x_2 = \bar{x}_{2,\varepsilon} = 0. \tag{131}$$

In summary, along the continuum, one has $\varepsilon \leq 1$, and:

$$\begin{cases} \bar{\mathbf{x}}_\varepsilon = (\sqrt{1-\varepsilon^2}, 0, 0, \varepsilon) \\ \phi_A(\varepsilon) = f_A(\bar{\mathbf{x}}_\varepsilon) = 2 - \sqrt{1-\varepsilon^2} = 1 + \dfrac{\varepsilon^2}{2} + O(\varepsilon^4) \\ \phi_A^+(\varepsilon) = f_A^+(\bar{\mathbf{x}}_\varepsilon) = 6 - 5\sqrt{1-\varepsilon^2} = 1 + \dfrac{5\varepsilon^2}{2} + O(\varepsilon^4) \\ \phi_B(\varepsilon) = f_B(\bar{\mathbf{x}}_\varepsilon) = (1-\varepsilon)^2 + \dfrac{9}{25}(1-\sqrt{1-\varepsilon^2}) = 1 - 2\varepsilon + \dfrac{59}{50}\varepsilon^2 + O(\varepsilon^4) \\ \phi_2(\varepsilon) = f_2(\bar{\mathbf{x}}_\varepsilon) = (1-\varepsilon)^2 + \dfrac{1}{5}(1-\sqrt{1-\varepsilon^2}) = 1 - 2\varepsilon + \dfrac{11}{10}\varepsilon^2 + O(\varepsilon^4) \\ \phi_3(\varepsilon) = f_3(\bar{\mathbf{x}}_\varepsilon) = (1-\varepsilon)^2 + (1-\sqrt{1-\varepsilon^2}) = 1 - 2\varepsilon + \dfrac{3}{2}\varepsilon^2 + O(\varepsilon^4) \end{cases} \tag{132}$$

As ε varies along the continuum, we observe the following facts that confirm our theoretical findings:

- $\mathbf{u} = O(\varepsilon^2)$ and $\mathbf{v} = O(\varepsilon)$;
- $\phi_1(0) = \phi_2(0) = \phi_3(0) = 1$;
- $\phi_1'(0) = 0$, $\phi_A(\varepsilon) = \phi_1(\varepsilon) = 1 + O(\varepsilon^2)$;
- $\phi_2'(0) = \phi_3'(0) = \phi_B'(0) = -\sigma_B = -2$.

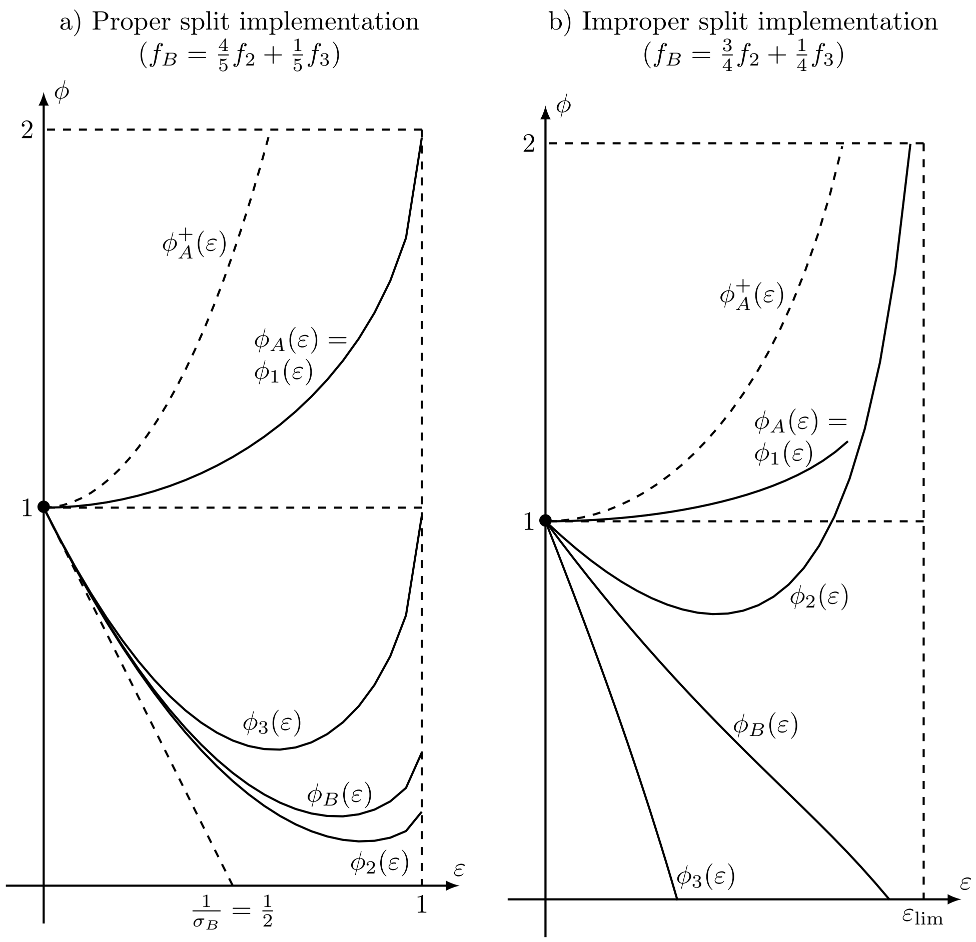

Figure 7: Test-case 4: One primary and two secondary cost functions

This test-case is illustrated on Figure 7a.

b) Improper split implementation

Let us now reconsider the above test-case, but with an improper definition of the cost function f_B obtained by modifying slightly the coefficients $\{\alpha_2, \alpha_3\}$; let us now set

$$f_B(\mathbf{x}) = \frac{3}{4}f_2(\mathbf{x}) + \frac{1}{4}f_3(\mathbf{x}) = -\frac{1}{4}(x_3-1)^2 + (x_4-1)^2 + \frac{1}{4} + \frac{2}{5}(1-x_1) \tag{133}$$

so that:

$$\begin{aligned} f_{AB}(\mathbf{x}) &= (1-\varepsilon)f_A^+(\mathbf{x}) + \varepsilon f_B(\mathbf{x}) \\ &= (1-\varepsilon)\left(5 + \sum_{i=1}^{4} x_i^2 - 5x_1\right) + \varepsilon\left[\frac{1-(x_3-1)^2}{4} + (x_4-1)^2 + \frac{2}{5}(1-x_1)\right] \end{aligned} \tag{134}$$

which should again be minimized w.r.t. (x_3, x_4) for fixed (x_1, x_2). This gives:

$$\begin{cases} 2(1-\varepsilon)x_3 - \dfrac{\varepsilon}{2}(x_3-1) = 0 \\ 2(1-\varepsilon)x_4 + 2\varepsilon(x_4-1) = 0 \end{cases} \tag{135}$$

and one gets:

$$x_3 = \bar{x}_{3,\varepsilon} = \frac{-\varepsilon}{4-5\varepsilon}, \quad x_4 = \bar{x}_{4,\varepsilon} = \varepsilon. \tag{136}$$

Then:

$$x_1 = \bar{x}_{1,\varepsilon} = \sqrt{1 - \varepsilon^2 \left[1 + \frac{1}{(4-5\varepsilon)^2}\right]}, \quad x_2 = \bar{x}_{2,\varepsilon} = 0. \tag{137}$$

Now the limitation on ε imposed by the radical in $\bar{x}_{1,\varepsilon}$ is more stringent

$$\varepsilon \leq \varepsilon_{\text{lim}} \approx 0.635\,. \tag{138}$$

These results are injected into the expressions of the cost functions and new graphs are plotted (see Figure 7b). In this setting, the derivatives $\phi_2'(0)$ and $\phi_3'(0)$ are not equal. Thus the minimization of f_B is not as effective on the two secondary cost functions f_2 and f_3. Evidently f_2 is not reduced as much.

6 Computations by meta-model-assisted software platform

When the first-order and second-order derivatives of the function are at hand, the numerical method proposed in Section 5 can be applied as such. In the more complex applications however, and in particular in PDE-constrained optimization, the first-order derivatives alone can be calculated either by adjoint formulations, automatic differentiation [11] or finite-differences. These techniques are not always trivial to implement and possibly costly. Hessians can also be calculated [16], but demanding a higher programming complexity and computational cost.

To circumvent these difficulties, we propose to apply the method to surrogate functions instead. To our experience, quadratic meta-models are adequate. However these are constructed globally, that is, elaborated once for all, for the cost functions, and locally, that is, constantly upgraded as new Nash equilibria are found, for the constraints. Indeed the numerical meta-model for the constraints must be locally accurate, since it strongly impacts the notion itself of Pareto stationarity. Relaxing the constraints would in effect alter, and likely unduly simplify, the multi-objective primary problem.

Are actually meta-modeled the two steering functions $f_A^+(\mathbf{x})$ and $f_B(\mathbf{x})$ and the constraints $\{c_k(\mathbf{x})\}$ $(k = 1, \ldots, K)$. The cost functions $\{f_j(\mathbf{x})\}$ $(j = 1, \ldots, M)$ are calculated exactly, according the problem specification. Gradients and diagonals of Hessians are locally approximated by central differencing using a small step size, and off-diagonal elements of Hessians, once for all, by more global least-squares in a box of larger size. For a complete description of the construction, see [8].

Denoting the meta-models by symbols with $\tilde{}$, this gives

$$\tilde{f}(\mathbf{x}) = f^\star + \left(\mathbf{x} - \mathbf{x}_A^\star, \nabla f^\star + \tfrac{1}{2}\mathbf{H}^\star(\mathbf{x} - \mathbf{x}_A^\star)\right) \tag{139}$$

for $f = f_A^+$ or f_B, and

$$\tilde{c}_k(\mathbf{x}) = c_k(\overline{\mathbf{x}}) + \left(\mathbf{x} - \overline{\mathbf{x}}, \overline{\nabla c_k} + \tfrac{1}{2}\overline{\mathbf{H}_{c_k}}(\mathbf{x} - \overline{\mathbf{x}})\right) \tag{140}$$

in which barred symbols are evaluated at $\overline{\mathbf{x}}$, that is, the just-computed previous Nash equilibrium point.

These meta-models being defined, the Nash-game equilibrium point for a given ε is found by a Schwarz-method-type iteration consisting in coordinating to convergence the following subproblems:

- **u-subproblem:** it is the minimization of a quadratic form subject to a set of quadratic constraints; the optimality conditions are solved by Newton's method.

- **v-subproblem:** it is the unconstrained minimization of a quadratic form; the optimality conditions are linear and solved by direct inversion.

These numerical procedures have been implemented in a generic code operating on two user-specified files defining the multi-objective problem and certain numerical parameters, (dimensions, accuracy tolerance, maximum allowable numbers of iterations, etc). Note that in this approximation the limit of convexity $\varepsilon_{\max}$ is known a priori [8]. Once the discrete continuum of Nash equilibria is completely determined, graphics are automatically elaborated by the process. The Inria software platform `http://mgda.inria.fr` [12] is currently being remodeled to permit a distant utilizer to execute the code, upon approval of a registration procedure, and receive output files including a textual report of the execution and automatically-generated graphics.

To validate our software platform, a number of test-cases have been computed [8] and firstly Test-case TC4 of Subsection 5.2. The computed variation with ε of the 4 variables is reproduced in Figure 8 for comparison with the theoretical result of (132) and Figure 7a. The match is excellent in spite of the derivative singularity of $x_1(\varepsilon) = \sqrt{1-\varepsilon^2}$ at $\varepsilon = 1$.

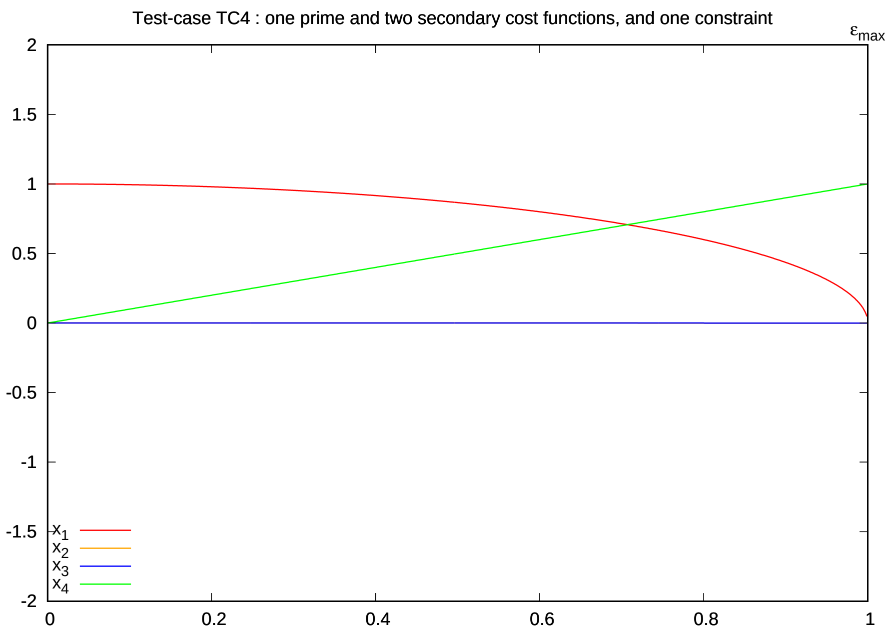

Figure 8: Test-case TC4. Variation with ε of the four variables as computed by the software platform.

Other test-cases. The following test-cases have been solved using the platform and documented in [8]:

- A variation of the classical Fonseca-Fleming test-case in R^5, involving two primary cost functions, two constraints, and one secondary cost function. In this test-case, the primary Pareto front (in function space) is concave over a large portion, indicating that the cost functions are not convex, but this is fixed by the procedure.
- A test-case in structural mechanics to optimize the sizing of a sandwich panel w.r.t. two failure forces.
- A test-case to optimize the flight performance of an aircraft (mass at take-off, range, approach speed, take-off distance) through 15 sizing parameters according to the classical Breguet laws.

7 Conclusion

Two complementary multi-objective optimization methods have been presented in a unified way:

– The MGDA, which is a generalization of the classical steepest-descent method to the multi-objective context, and that permits rapid convergence from a starting point to a Pareto-stationary point provided that the gradients are known or accurately approximated; in the quasi-Riemannian formulation [7] constraints are handled;

– The Nash-MGDA approach for prioritized optimization by Nash games that permit from a point on the Pareto front associated with the primary cost functions to generate a path which is, in function space, tangent to the front.

In this way the Pareto front can be explored by alternating both methods over finite segments.

Several test-cases have been treated successfully in [8], some of which partly reproduced here for illustration. In a test-case of parameter sizing for the optimization of the flight performance of an aircraft, the prioritized optimization method complemented an evolutionary strategy (PAES) to identify the Pareto front and it revealed more easily handled, less subject to scaling and ill-conditioning, and far less costly than PAES itself.

The numerical procedures used in our development have been described precisely in [8], and our numerical code can be executed by other utilizers after acceptance of a registration. The related information is provided in the Inria software platform `http://mgda.inria.fr` which is being reorganized accordind to three headings:

1. **MGDA**, for the calculation of a descent direction common to an arbitrary set of parametric gradients;

2. **Quasi-Riemannian approach**, to handle constrained problems, recommendations and examples;

3. **Nash-MGDA platform**, for prioritized multi-objective optimization by Nash games.

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References

[1] J.-P. Aubin, *Mathematical methods of game and economic theory*, Courier Corporation, 2007.

[2] S. P. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press, 2004, ISBN 978-0-521-83378-3. Retrieved October 3, 2011.

[3] L. Cambier and J.-P. Veuillot, *Status of the elsA CFD software for flow simulation and multidisciplinary applications*, AIAA Paper 2008-664, 2008.

[4] J.-A. Désidéri, *Multiple-gradient descent algorithm (MGDA) for multiobjective optimization*, Comptes Rendus de l'Académie des Sciences Paris **350** (2012), 313–318, `http://dx.doi.org/10.1016/j.crma.2012.03.014`.

[5] ———, *Numerical methods for differential equations, optimization, and technological problems*, Modeling, Simulation and Optimization for Science and Technology, Fitzgibbon, W.; Kuznetsov, Y.A.; Neittaanmäki, P.; Pironneau, O. Eds., vol. 34, ch. Multiple-Gradient Descent Algorithm (*MGDA*) for Pareto-Front Identification, Springer-Verlag, 2014, J. Périaux and R. Glowinski Jubilees.

[6] ———, *Révision de l'algorithme de descente à gradients multiples (MGDA) par orthogonalisation hiérarchique*, Research Report 8710, INRIA, April 2015, `https://hal.inria.fr/hal-01139994`.

[7] ———, *Quasi-riemannian multiple gradient descent algorithm for constrained multiobjective differential optimization*, Research Report 9159, INRIA, 21 March 2018, `https://hal.inria.fr/hal-01740075v1`.

[8] ———, *Platform for prioritized multi-objective optimization by metamodel-assisted Nash games*, Research Report 9290, INRIA, September 2019.

[9] J.-A. Désidéri and R. Duvigneau, *Parametric optimization of pulsating jets in unsteady flow by Multiple-Gradient Descent Algorithm (MGDA)*, Numerical Methods for Differential Equations, Optimization, and Technological Problems, Modeling, Simulation and Optimization for Science and Technology (J. Périaux, W. Fitzgibbon, B. Chetverushkin, and O. Pironneau, eds.), Jan 2017, `https://hal.inria.fr/hal-01414741v1`.

[10] J.-A. Désidéri, R. Duvigneau, and A. Habbal, *Computational intelligence in aerospace sciences, v. m. becerra and m. vassile eds.*, Progress in Astronautics and Aeronautics, T. C. Lieuwen Ed.-in-Chief, vol. 244, ch. Multi-Objective Design Optimization Using Nash Games, American Institute for Aeronautics and Astronautics Inc., Reston, Virginia, 2014.

[11] Inria Project Team Ecuador, *Tapenade, version 3*, `https://www-sop.inria.fr/tropics/`.

[12] J.-A. Désidéri *et al*, *MGDA Platform, Multiple Gradient Descent Algorithm for Multi-Objective Differentiable Optimization*, `http://mgda.inria.fr`.

[13] P. E. Gill, W. Murray, and M. H Wright, *Practical optimization*, twelfth printing 2000 ed., Academic Press, New York London, 1986.

[14] E. Roca León, *Aeromechanical simulations for the optimization of helicopter rotors in forward flight*, Ph.D. thesis, Université Nice Sophia-Antipolis, 2014, `https://www.theses.fr/2014NICE4076`.

[15] B. Abou El Majd, *Hierarchical algorithms and game strategies for multidisciplinary optimization: application to the wing shape optimization of a business jet*, Ph.D. thesis, Université Nice Sophia-Antipolis, 2007, `https://www.theses.fr/2007NICE4073`.

[16] M. Martinelli and R. Duvigneau, *On the use of second-order derivatives and metamodel-based Monte-Carlo for uncertainty estimation in aerodynamics*, Computer & Fluids **37** (2010), no. 6.

[17] Q. Mercier, F. Poirion, and J.-A. Désidéri, *A stochastic multiple gradient descent algorithm*, European Journal of Operational Research, ELSEVIER Publish. **271** (2018), no. 3, 808–817.

[18] A. Minelli, *Aero-acoustic shape optimization of a supersonic business jet*, Ph.D. thesis, Université Nice Sophia-Antipolis, 2013, `https://www.theses.fr/2013NICE4107`.

[19] F. Poirion, Q. Mercier, and J.-A. Désidéri, *Descent algorithm for nonsmooth stochastic multi-objective optimization*, Computational Optimization and Applications **68** (2017), no. 2, 317–331.

[20] Q.Mercier, F.Poirion, and J.-A. Désidéri, *Non-convex multiobjective optimization under uncertainty: a descent algorithm. application to sandwich plate design and reliability*, Engineering Optimization **0** (2018), no. 0, 1–20.

[21] Z. Tang, J.-A. Désidéri, and J. Périaux, *Multi-criterion aerodynamic shape-design optimization and inverse problems using control theory and Nash games*, Journal of Optimization Theory and Applications **135** (2007), no. 1.

[22] J. H. Wilkinson, *The algebraic eigenvalue problem (numerical mathematics and scientific computation)*, Oxford Science Publications, 1965 (first edition).

A Stability of eigenvectors w.r.t numerical inaccuracies

Let $\mathbf{A}_0$ be a baseline-case matrix and $\delta\mathbf{B}$ a perturbation matrix applied to it, where δ is a small positive number representative of the computational accuracy. Both matrices are $n \times n$ and real-symmetric, and $\mathbf{A}_0$ is assumed to be diagonal for simplicity. The set of eigenvectors of $\mathbf{A}_0$, $\{\mathbf{e}_j\}$ ($j = 1, \dots, n$), constitutes the canonical basis ($e_{i,j} = \delta_{i,j}$, Krönecker symbol). The set of eigenvectors of $\mathbf{A}_0 + \delta\mathbf{B}$ is a perturbation of it, denoted $\{\mathbf{e}'_j\}$.

Given the dimension p, the territory splitting of R^n should be made into the supplementary subspaces

$$U = Sp\big(\mathbf{e}_1, \dots, \mathbf{e}_{n-p}\big), \quad V = Sp\big(\mathbf{e}_{n-p+1}, \dots, \mathbf{e}_n\big), \tag{141}$$

but instead, it is done into:

$$U' = Sp\big(\mathbf{e}'_1, \dots, \mathbf{e}'_{n-p}\big), \quad V' = Sp\big(\mathbf{e}'_{n-p+1}, \dots, \mathbf{e}'_n\big). \tag{142}$$

A given vector $\mathbf{x}$ is not split as it should into the subvectors $(\mathbf{x_u}, \mathbf{x_v})$, but instead into $(\mathbf{x'_u}, \mathbf{x'_v})$:

$$\mathbf{x} = \mathbf{x_u} + \mathbf{x_v} = (\mathbf{I}_n - \mathbf{P})\mathbf{x} + \mathbf{P}\mathbf{x} = \mathbf{x'_u} + \mathbf{x'_v} = (\mathbf{I}_n - \mathbf{P}')\mathbf{x} + \mathbf{P}'\mathbf{x}. \tag{143}$$

Clearly, a measure of the projection error on $\mathbf{x}$ is given by

$$E(\mathbf{x}) = \|\mathbf{x'_u} - \mathbf{x_u}\| = \|\mathbf{x'_v} - \mathbf{x_v}\| = \|(\mathbf{P}' - \mathbf{P})\mathbf{x}\| = \|(\mathbf{Q}' - \mathbf{Q})\mathbf{x}\| \tag{144}$$

where $\mathbf{Q}$ is the diagonal matrix $\mathbf{I}_n - \mathbf{P} = \mathsf{Diag}(1, \dots, 1, 0, \dots, 0)$ with $n - p$ ones, and $\mathbf{Q}'$ the corresponding projection matrix after perturbation. Thus, we propose to assess the stability of the eigenvector split by the following index:

$$s = \frac{\|\mathbf{Q}' - \mathbf{Q}\|_\infty}{\delta}. \tag{145}$$

The matrix $\mathbf{Q}'$ is the matrix associated with the orthogonal projection of the canonical basis onto the span of $\{\mathbf{e}'_1, \dots, \mathbf{e}'_{n-p}\}$. Hence, for a given index j:

$$\mathbf{Q}'\mathbf{e}_j = \sum_{k=1}^{n-p} (\mathbf{e}_j . \mathbf{e}'_k)\, \mathbf{e}'_k = \sum_{k=1}^{n-p} (\mathbf{e}'_{j,k})\, \mathbf{e}'_k = \sum_{k=1}^{n-p} (\mathbf{e}'_{j,k}) \sum_{i=1}^{n} (\mathbf{e}'_{i,k})\, \mathbf{e}_i = \sum_{i=1}^{n} \mathbf{Q}'_{i,j}\, \mathbf{e}_i \tag{146}$$

where the generic element of the matrix $\mathbf{Q}'$ is given by:

$$\mathbf{Q}'_{i,j} = \sum_{k=1}^{n-p} \mathbf{e}'_{i,k} \mathbf{e}'_{j,k}. \tag{147}$$

A numerical experiment was conducted in which $n = 10$, and $\mathbf{A}_0 = \mathsf{Diag}(1, 2, \dots, 10)$. A perturbation matrix was calculated using a symmetric matrix $\mathbf{B}$ whose elements were defined by random drawal in [-1,1] and $\delta = 10^{-5}$. The eigenvectors $\{\mathbf{e}'_j\}$ were calculated using the procedure DSYEV of the LAPACK library, permitting to compute the matrix $\mathbf{Q}'$, and the index s from (145). Firstly, after setting $p = 5$, we found the index $s \doteq 1.4$, confirming, unsurprisingly, that the eigenvector problem related to $\mathbf{A}_0$ was very well conditioned. The experiment was then repeated after replacing $\mathbf{A}_0$ by the diagonal matrix $\mathbf{A}_1$ differing from $\mathbf{A}_0$ by the sole 5-fth eigenvalue, increased from 5 to 5.99 to be close to the next larger eigenvalue 6. Note that δ that sizes the perturbation was set three orders of magnitude smaller than the distance between the closest two eigenvalues (0.01). In spite of that, with the same $p = 5$, the index raised nearly two orders of magnitude to 87.2. Then, applying the perturbation again to the same matrix $\mathbf{A}_1$, the value of p was changed. With the smaller value $p = 4$, $s \doteq 1.7$ was found, and with the larger value $p = 6$, $s \doteq 1.2$. This experiment tends to confirm that for the eigenvector problem to be stable, the split should be made by grouping modes in two subsets associated with well-separated corresponding sets of eigenvalues. In other words, if at all possible, one should avoid choosing a value of p that places the cut-off in a dense region of the spectrum. However within each group, dense zones seem to be harmless.

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